Date : ....

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™ (INSTITUTE OF MATHEMATICAL SCIENCES)

MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

Under the guidance *of* K. Venkanna

MATHEMATICS

PAPER - II : MODERN ALGEBRA, REAL ANALYSIS, COMPLEX & LPP

TEST CODE: TEST-2: IAS(M)/16-JUNE.-2019

Time: 3 Hours **Maximum Marks: 250**

INSTRUCTIONS

1. This question paper-cum-answer booklet has **50** pages and has **33 PART/SUBPART** questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name

Roll No.

Test Centre

Medium

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

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INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			
	(b)			
	(c)			
	(d)			
4	(a)			
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	(d)			
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	(b)			
	(c)			
	(d)			
8	(a)			
	(b)			
	(c)			
	(d)			
			Total Marks	

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SECTION – A

1. (a) Are the following groups cyclic groups ? Give reasons.

(i) The Klein 4-group.

(ii) The dihedral group D_4.

(iii) The group of all roots (real or complex) of the equation $x^n - 1 = 0$.

(iv) The group Q^* of non-zero rationals, for multiplication. **[10]**

1. (b) Let R be a commutative ring with unit element whose only ideals are (0) and R itself. Then R is a field. **[10]**

1. (c) For $u_1 > 0$, the sequence u_n defined by

$$u_{n+1} = 1 + \frac{1}{u_n} \forall n, \text{ converges to} \left(\frac{\sqrt{5}+1}{2}\right)$$ **(10)**

1. (d) Prove that the function $u = e^{-x}$ (x cos y + y sin y). is harmonic and find the corresponding analytic function. **(10)**

1. (e) Write the dual of the problem.

Min $Z = 2x_2 + 5x_3$

Subject to
$x_1 + x_2 \geq 2$
$2x_1 + x_2 + 6x_3 \leq 6$
$x_1 - x_2 + 3x_3 = 4$
$x_1, x_2, x_3 \geq 0.$ **[10]**

2. (a) (i) Show that every subgroup of an abelian group is normal.

(ii) Is the converse of Problem (i) true ? If yes, prove it, if no, give an example of a non-abelian group all of whose subgroups are normal. **[12]**

2. (b) Suppose that N and M are two normal subgroups of G and that $N \cap M = (e)$. show that for any $n \in N$, $m \in M$, $nm = mn$. **[08]**

2. (c) Test the convergence of

(i) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

(ii) Prove that the integral $\int_0^\infty x^{m-1}e^{-x}dx$ is convergent if and only if m > 0. **[15]**

2. (d) Apply the method of contour integration to prove that

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \frac{\pi}{6}.$$ **[15]**

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3. (a) Prove, by an example, that we can find three groups $E \subset F \subset G$, where E is normal in F, F is normal in G, but E is not normal in G.

[10]

3. (b) Given an example of an integral domain which has an infinite number of elements, yet is of finite characteristic. **[10]**

3. (c) Suppose that f is defined on $I = \{x : 0 \le x \le 1\}$ by the formula

$$f(x) = 0\begin{cases} \dfrac{1}{2^n} & \text{if } x = \dfrac{j}{2^n} \text{ where } j \text{ is an odd integer} \\ & \text{and } 0 < j < 2^n, n = 1, 2, \ldots\ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Determine whether or not f is integrable and prove your result. **[15]**

3. (d) Using simplex method solve the L.P. problem.

Mini $Z = 4x_1 + 8x_2 + 3x_3$

subject to $x_1 + x_2 \geq 2$

$2x_1 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0.$ **[15]**

4. (a) Let M_1 and M_2 be distinct maximal ideals of R. Show that $M_1 + M_2 = R$. Define a suitable ring homomorphism of R onto $R/M_1 \times R/M_2$ and deduce that $R/M_1 \cap M_2 \simeq R/M_2 \times R/M_2$. **[15]**

4. (b) The function $f(x) = 1/x$ is continuous on $(0, 1)$ but not uniformly continuous. **[13]**

4. (c) (i) Specify the nature of singularity at $z = -2$ of

$$f(z) = (z-3)\sin\frac{1}{z+2}$$

(ii) Find zeros and poles of $\left(\dfrac{z+1}{z^2+1}\right)^2$ **[12]**

4. (d) An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of manhours that would be required for each job-man combination. This is given in the matrix form in adjacent table. Find the optimum assignment that will result in minimum manhours needed.

Man \ Job	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

[10]

SECTION – B

5. (a) Give an example of a group which is not a cyclic group, but every proper subgroup of which is cyclic. **[10]**

5. (b) Consider the following rings R and R′ with four elements

$R = \{a, b, c, d\}$ with + and • defined by

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

•	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

and $R' = \{x, y, z, \mathbf{t}\}$ with + and • defined by

+	x	y	z	t
x	x	y	z	t
y	y	z	t	x
z	z	t	x	y
t	t	x	y	z

•	x	y	z	t
x	x	x	x	x
y	x	y	z	t
z	x	z	x	z
t	x	t	z	y

Investigate whether R and R′ are isomorphic. **[10]**

5. (c) The sequence $nx/(1 + n^2 x^2)$ is not uniformly convergent over **R** but it is uniformly convergent on $\{x : |x| > k > 0\}$. **[10]**

5. (d) If $f(z)=\dfrac{x^3y(y-ix)}{x^6+y^2}, z\neq 0$ and f(0) = 0, show that $\dfrac{f(z)-f(0)}{z}\to 0 \text{ as } z\to 0$ along any radius vector but not as $z\to 0$ in any manner. **[10]**

5. (e) A firm plans to purchase at least 200 quintals of scrap containing high quality metal X and low quality metal. Y. It decides that the scrap to be purchased must contain at least 100 quintals of X-metal and not more than 35 quintals of Y-metal. The firm can purchase the scrap from two suppliers (A and B) in unlimited quantities. The percentage of X and Y metals in terms of weight in the scraps supplied by A and B is given below :

Metals	supplier A	Supplier B
X	25%	75%
Y	10%	20%

The price of A's scrap is Rs. 200 per quintal and that of B's is Rs. 400 per quintal. Formulate this problem as LP model and solve it graphically to determine the quantities that the firm should buy from the two suppliers so as to minimize total purchase cost. **[10]**

6. (a) (i) Let G be the group of all 2 × 2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo p, p a prime number, such that $ad - bc \neq 0$. G forms a group relative to matrix multiplication. What is $o(G)$?

(ii) Let H be the subgroup of the G of part (a) defined by

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \,\middle|\, ad - bc = 1 \right\}.$$

What is $o(H)$? **[15]**

6. (b) Investigate whether the following ideals are (i) prime ideals, (ii) maximal ideals in the ring R.

(i) $R = Z_6 \text{ and } I = \{\bar{0}, \bar{2}, \bar{4}\}$.

(ii) R = 2Z and I = 4Z

(iii) R = C[0, 1] the ring of continuous functions on [0, 1] and $I = \left\{x \mid x\left(\frac{1}{2}\right) = 0\right\}$. **[15]**

6. (c) Prove that every integral domain can be imbedded in a field. **[20]**

7. (a) Prove that the following sets are bounded

$\{n^{1/n} : n \in N\}, \left\{\left(1+\frac{1}{n}\right)^n : n \in N\right\}, \{a^{1/n} : a > 0 \text{ and } n \in N\}.$

Give supremum and infimum of each of these sets. **[08]**

7. (b) Show that $\prod_{1}^{\infty}\left(1-\frac{1}{4n^2}\right)$ converges and its limit lies between $\frac{1}{2}$ and 1. **[10]**

7. (c) For x > – 1, test for convergence

$$1+\frac{2^2}{3.4}x+\frac{2^2.4^2}{3.4.5.6}x^2+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^3+....$$ **[15]**

7. (d) Let a function $f : \mathbf{R} \to \mathbf{R}$ satisfy the equation

$f(x + y) = f(x) + f(y), \forall\, x, y \in \mathbf{R}$. Show that

(i) If f is continuous at the point $x = a$, then it is continuous for all $x \in \mathbf{R}$.

(ii) If f is continuous then $f(x) = kx$, for some constant k. **[15]**

8. (a) Using Cauchy's/Cauchy's integral formula.

(i) Evaluate $\oint_C \frac{\sin 3z}{z+\pi/2} dz$ if c is the circle $|z| = 5$.

(ii) Evaluate $\oint_C \frac{e^{3z}}{z-\pi i} dz$ if C is : (A) the circle $|z-1| = 4$, (B) the ellipse $|z-2| + |z + 2| = 6$. **[10]**

8. (b) Find the Laurent expansion of $\frac{z}{(z+1)(z+2)}$ about the singularity $z = -2$. Specify the region of convergence. **[07]**

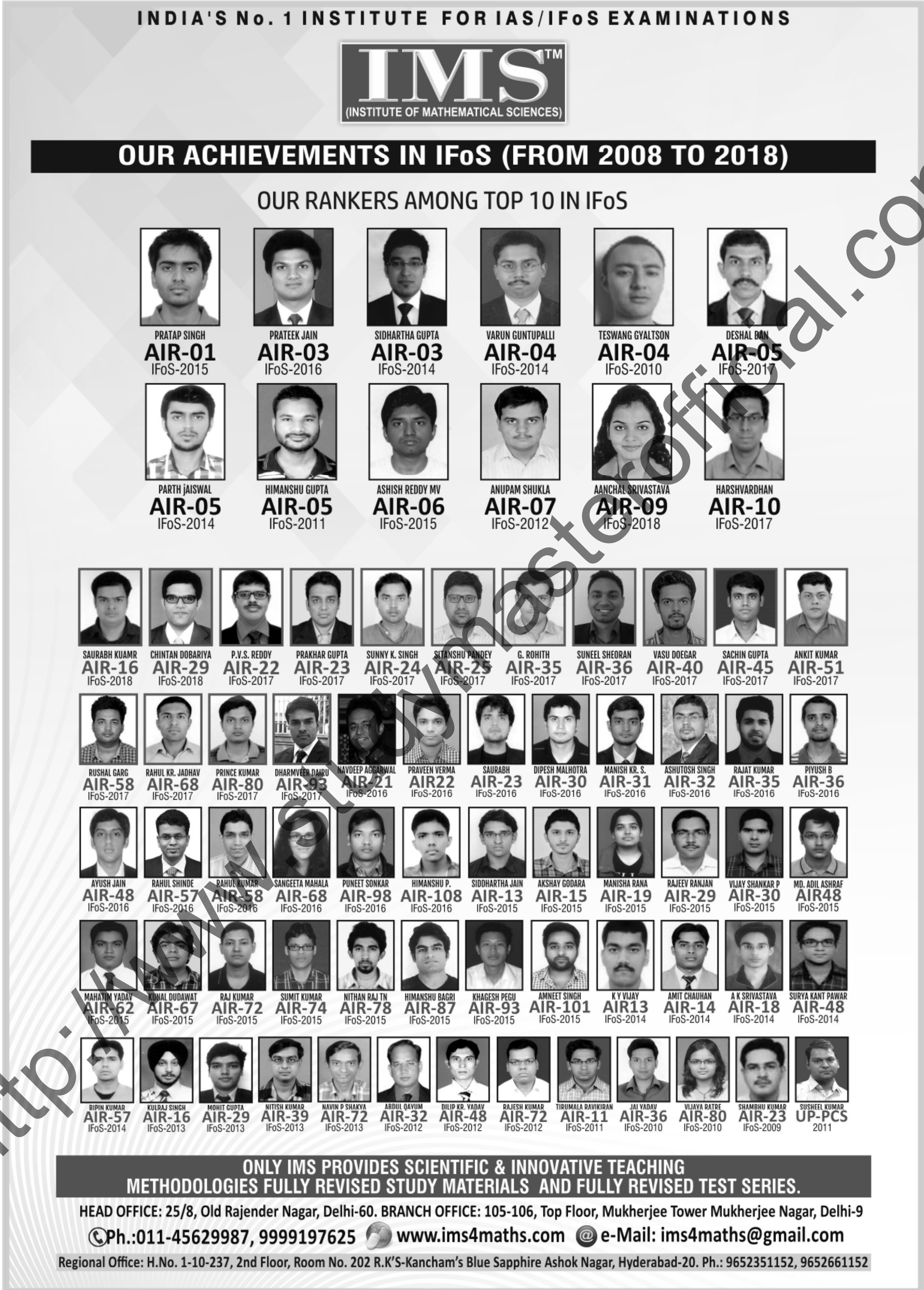
8. (c) By integrating $e^{iz} / (z - ai)$, $(a > 0)$ round a suitable contour, prove that

$$\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2 + a^2} dx = 2\pi e^{-a}.$$ **[13]**

8. (d) Solve the following transportation problem **[20]**

	To			Available
From	2	7	4	5
	3	3	1	8
	5	4	7	7
	1	6	2	14
Required	7	9	18	34

ROUGH SPACE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MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

MATHEMATICS

Under the guidance *of* **K. Venkanna**

TEST CODE: TEST-2: IAS(M)/16-JUNE.-2019

PAPER - II : MODERN ALGEBRA, REAL ANALYSIS, COMPLEX & LPP

Time: 3 Hours | **Maximum Marks: 250**

INSTRUCTIONS

Each question is printed only in English.

Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.

The number of marks carried by each question is indicated at the end of the question.

Assume suitable data if considered necessary and indicate the same clearly.

Symbols/notations carry their usual meanings, unless otherwise indicated.

All questions carry equal marks.

Important Note: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

IMS
INSTITUTE FOR IAS/IFOS EXAMINATIONS

HEAD OFFICE:25/8, OLD RAJINDER NAGAR MARKET, DELHI-60. Ph.: 09999197625, 011-45629987
BRANCH OFFICE: 105-106, Top Floor, Mukherjee Tower, Mukherjee Nagar, Delhi-9
REGIONAL OFFICE : 1-10-237, IInd Floor, Room No. 202 R.K'S Kancham's Blue Sapphire Ashok Nagar Hyderabad-20.
Mobile No.: 09652351152, 09652661152
www.ims4maths.com || Email: ims4maths@gmail.com

SECTION – A

1. (a) Are the following groups cyclic groups ? Give reasons.

(i) The Klein 4-group.

(ii) The dihedral group D_4.

(iii) The group of all roots (real or complex) of the equation $x^n - 1 = 0$.

(iv) The group Q* of non-zero rationals, for multiplication. **[10]**

1. (b) Let R be a commutative ring with unit element whose only ideals are (0) and R itself. Then R is a field. **[10]**

1. (c) For $u_1 > 0$, the sequence u_n defined by

$$u_{n+1} = 1 + \frac{1}{u_n} \forall n, \text{ converges to } \left(\frac{\sqrt{5}+1}{2}\right).$$ **(10)**

1. (d) Prove that the function $u = e^{-x}(x \cos y + y \sin y)$. is harmonic and find the corresponding analytic function. **(10)**

1. (e) Write the dual of the problem.

Min $Z = 2x_2 + 5x_3$

Subject to $x_1 + x_2 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 4$

$x_1, x_2, x_3 \geq 0.$ **[10]**

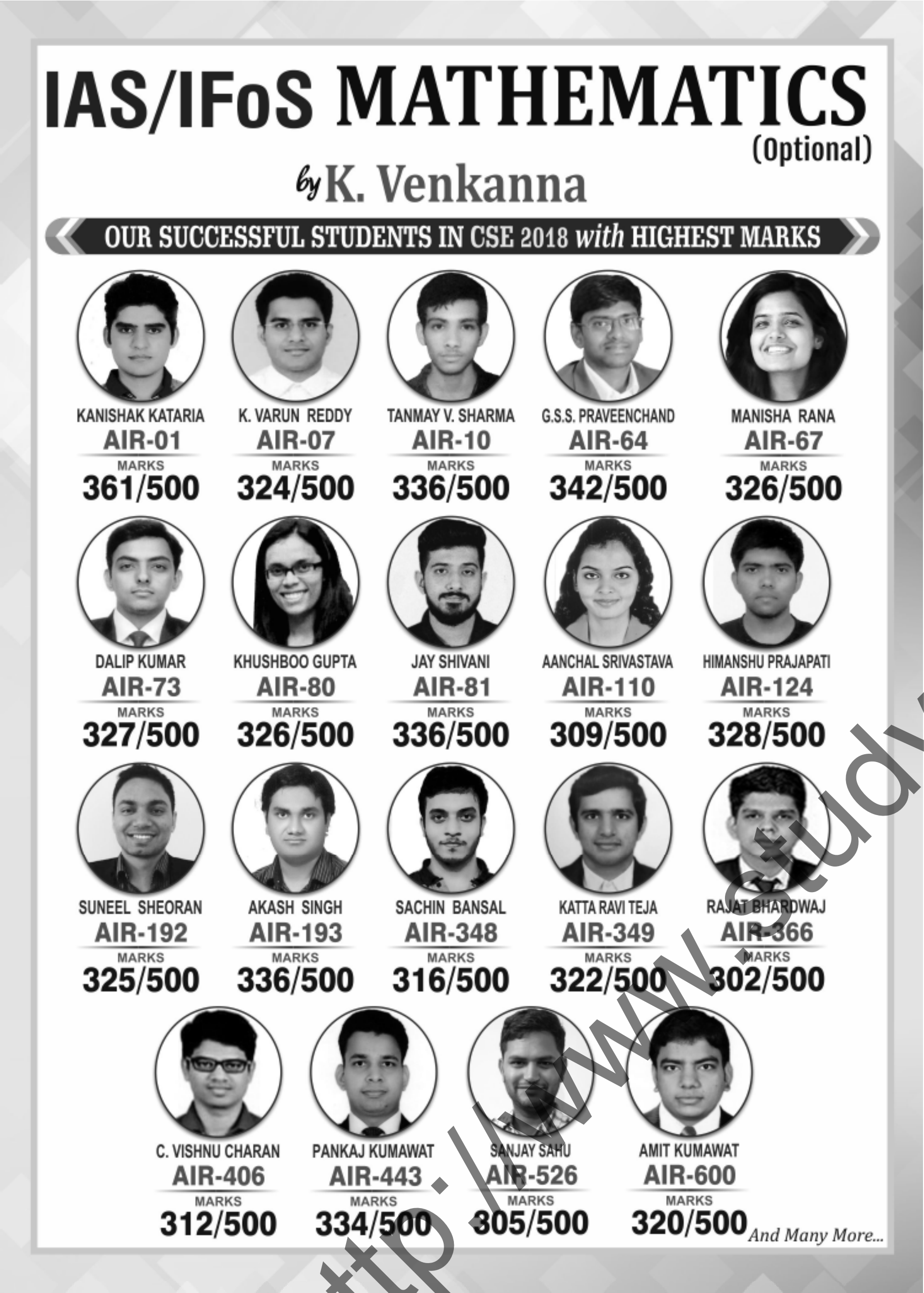
2. (a) (i) Show that every subgroup of an abelian group is normal.

(ii) Is the converse of Problem (i) true ? If yes, prove it, if no, give an example of a non-abelian group all of whose subgroups are normal. **[12]**

2. (b) Suppose that N and M are two normal subgroups of G and that $N \cap M = (e)$. show that for any $n \in N$, $m \in M$, $nm = mn$. **[08]**

2. (c) Test the convergence of

(i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^3}}$



(ii) Prove that the integral $\int_0^\infty x^{m-1}e^{-x}dx$ is convergent if and only if $m > 0$. **[15]**

2. (d) Apply the method of contour integration to prove that

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}d\theta = \frac{\pi}{6}.$$ **[15]**

3. (a) Prove, by an example, that we can find three groups $E \subset F \subset G$, where E is normal in F, F is normal in G, but E is not normal in G. **[10]**

3. (b) Given an example of an integral domain which has an infinite number of elements, yet is of finite characteristic. **[10]**

3. (c) Suppose that f is defined on $I = \{x : 0 \le x \le 1\}$ by the formula

$$f(x) = 0\begin{cases} \dfrac{1}{2^n} & \text{if } x = \dfrac{j}{2^n} \text{ where } j \text{ is an odd integer} \\ & \text{and } 0 < j < 2^n, n = 1, 2, \ldots.., \\ 0 & \text{otherwise.} \end{cases}$$

Determine whether or not f is integrable and prove your result. **[15]**

3. (d) Using simplex method solve the L.P. problem.

Mini $\quad Z = 4x_1 + 8x_2 + 3x_3$

subject to $\quad x_1 + x_2 \ge 2$

$2x_1 + x_3 \ge 5$

$x_1, x_2, x_3 \ge 0.$ **[15]**

4. (a) Let M_1 and M_2 be distinct maximal ideals of R. Show that $M_1 + M_2 = R$. Define a suitable ring homomorphism of R onto $R/M_1 \times R/M_2$ and deduce that $R/M_1 \cap M_2 \simeq R/M_2 \times R/M_2$. **[15]**

4. (b) The function $f(x) = 1/x$ is continuous on (0, 1) but not uniformly continuous. **[13]**

4. (c) (i) Specify the nature of singularity at $z = -2$ of

$$f(z) = (z-3)\sin\frac{1}{z+2}$$

(ii) Find zeros and poles of $\left(\frac{z+1}{z^2+1}\right)^2$ **[12]**

4. (d) An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of manhours that would be required for each job-man combination. This is given in the matrix form in adjacent table. Find the optimum assignment that will result in minimum manhours needed.

Man \ Job	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

[10]

SECTION – B

5. (a) Give an example of a group which is not a cyclic group, but every proper subgroup of which is cyclic. **[10]**

5. (b) Consider the following rings R and R′ with four elements

$R = \{a, b, c, d\}$ with $+$ and $\bullet$ defined by

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

•	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

Anyone who has done B.Tech / M.Tech / B.Sc / M.Sc and has an interest in Maths. Usually commit and their mitigation measures. For example, I commit a lot of mistakes when doing Integration by parts and usually the error involves missing negative (-) sign etc. Therefore whenever I come across such type of question I try to devote extra 1 minute to re-check all my steps.

Maths.stackexchange.com is the best online resource for preparation. You can create an account and get your maths questions answered within minutes.

Why did I score only 262?

Among all the students in the final list who had Maths as an optional, I have scored the least. My paper - 1 was a complete disaster and I only scored 92 marks in it. In fact I could only attempt 160 marks paper and had to leave 90 marks paper completely.

The reasons for the above situation in Paper - 1 are as follows:

1. **Lack of written practice:** In many topics (especially statics and dynamics) I used to just look at a question and its solution without solving it first. As a result I forgot the exact method in the exam hall!
2. **Left many topics:** I prepared only 25% 3-D, 80% Calculus and 25% Statics & Dynamics and had to pay a heavy price in the exam.

On the other hand my preparation for paper - 2 was excellent and therefore I scored an amazing 170 marks in it

BHAVESH MISHRA
AIR-58 in CSE-2014

Easy paper: The difficulty level of paper is quite moderate and almost all questions are directly picked from the IMS Test Series / Standard Textbooks.

WHO SHOULD TAKE IT?

Myths around science subjects.

Coaching institutions have mastered the art of brainwashing students and creating an atmosphere of gloom and doom around science subjects. There are lots of myths circulating among students. Let's bust these myths.

1. **Maths optional is only for students from IITs: Definitely not.** Anyone willing to put in hard work can easily score very high marks. The best example being **Nitish K (Rank 8) who is not from any IIT**.
2. **There is heavy scaling:** Let the data speak for itself. I attempted 240 marks in Paper 2 and got 170 marks. Now would you call it a scaling?
3. **It plays no role in GS:** Yes it's true that science optional subjects don't overlap with GS but it's equally true that GS has never been a rank decider in UPSC IAS.
4. **There are 3 major things that decides your rank:** Essay, Optional and Interview. Even if one puts in 5 years of efforts in GS the advantage in terms of marks would be around 30 marks or so but 1 year of dedicated effort in maths would give you 50+ marks advantage straightaway.

Do's and Dont's:

1. Practice, Practice and Practice. The key to success in maths is filling up as many notebooks as you can, during the preparation stage. The more you sweat during preparation the less you will bleed in the battlefield!
2. Don't read Maths book / notes like GS. It is a recipe for disaster. Rather always study with pen, paper and calculator.
3. While solving examples don't jump to see solution first. Try giving your best shot and after making sure that you are not able to solve it using your present knowledge then only look at the answer. This will ensure that better retention.
4. Generally we make lots of silly mistakes while solving a question. It is best to catch these errors early and not repeat them in exam hall. The best strategy for this is to maintain a notebook of errors that you

and $R' = \{x, y, z, \mathbf{t}\}$ with + and • defined by

+	x	y	z	t
x	x	y	z	t
y	y	z	t	x
z	z	t	x	y
t	t	x	y	z

•	x	y	z	t
x	x	x	x	x
y	x	y	z	t
z	x	z	x	z
t	x	t	z	y

Investigate whether R and R′ are isomorphic. **[10]**

5. (c) The sequence $nx/(1 + n^2 x^2)$ is not uniformly convergent over **R** but it is uniformly convergent on $\{x : |x| > k > 0\}$. **[10]**

5. (d) If $f(z) = \dfrac{x^3 y(y - ix)}{x^6 + y^2}, z \neq 0$ and $f(0) = 0$, show that

$\dfrac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner. **[10]**

5. (e) A firm plans to purchase at least 200 quintals of scrap containing high quality metal X and low quality metal. Y. It decides that the scrap to be purchased must contain at least 100 quintals of X-metal and not more than 35 quintals of Y-metal. The firm can purchase the scrap from two suppliers (A and B) in unlimited quantities. The percentage of X and Y metals in terms of weight in the scraps supplied by A and B is given below :

Metals	supplier A	Supplier B
X	25%	75%
Y	10%	20%

The price of A's scrap is Rs. 200 per quintal and that of B's is Rs. 400 per quintal. Formulate this problem as LP model and solve it graphically to determine the quantities that the firm should buy from the two suppliers so as to minimize total purchase cost. **[10]**

6. (a) (i) Let G be the group of all 2 × 2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo p, p a prime number, such that $ad - bc \neq 0$. G forms a group relative to matrix multiplication. What is $o(G)$?

(ii) Let H be the subgroup of the G of part (a) defined by

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \middle| ad - bc = 1 \right\}.$$

What is $o(H)$? **[15]**

6. (b) Investigate whether the following ideals are (i) prime ideals, (ii) maximal ideals in the ring R.

(i) $R = Z_6$ and $I = \{\bar{0}, \bar{2}, \bar{4}\}$.

(ii) R = 2Z and I = 4Z

(iii) R = C[0, 1] the ring of continuous functions on [0, 1] and $I = \left\{ x \mid x\left(\frac{1}{2}\right) = 0 \right\}$. **[15]**

6. (c) Prove that every integral domain can be imbedded in a field. **[20]**

7. (a) Prove that the following sets are bounded

$\{n^{1/n} : n \in N\}, \left\{ \left(1 + \frac{1}{n}\right)^n : n \in N \right\}$, $\{a^{1/n} : a > 0$ and $n \in N\}$.

Give supremum and infimum of each of these sets. **[08]**

7. (b) Show that $\prod_{1}^{\infty} \left(1 - \frac{1}{4n^2}\right)$ converges and its limit lies between $\frac{1}{2}$ and 1. **[10]**

Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theoretical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants. All the best!

Bhavesh Mishra (AIR-58)

***in* IAS-2014 Examination**

CLASSROOM STUDENT

Why Maths?

Simply because it is the best performing optional subject in UPSC/IAS.

Extremely high scoring: If you get your maths optional right then you will make it to the final list. This year one of my batch mate in IMS **Nitish K (Rank 8)** has got a mind boggling 346 marks.

Certainty: If you have attempted your paper well then you are sure that you will get good marks. For example this year just by attempting 400 marks paper you could get a decent 260+ marks. Even if you don't get good marks in first attempt but you can be sure that you will increase your marks in subsequent attempt(s).

Fun: Mathematics is a delightful subject and therefore doing maths takes you away from somewhat boring humanities.

Good Impression: The fact that you have taken Maths makes a good impression on interview board members (it happened in my case !). They are very pleased to see that you have opted for a tough optional.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

TEST SERIES

Test series is very important for this optional. I had joined IMS test series which helped me in identifying my weak areas. In both CSE and IFoS mains, there were many questions similar to those covered in IMS test series. With enough practice, a candidate can achieve the ability to complete the maths paper in 3 hours. It is important to assess your performance after each test. Necessary steps should be taken to rectify common mistakes that you are commiting in the test series. You should be alert not to repeat the same mistakes again & again. As your performance improves with every test, the actual mains paper will seem just like any other test & you will be able to comfortably complete it. Presentation of your answer matters a lot. Your aim should be to make examiner's life as easy as possible so that he/she will award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter. Once you finish paper 1, don't start immediately analyzing your performance.

7. (c) For $x > -1$, test for convergence

$$1+\frac{2^2}{3.4}x+\frac{2^2.4^2}{3.4.5.6}x^2+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^3+....$$ **[15]**

7. (d) Let a function $f : \mathbf{R} \to \mathbf{R}$ satisfy the equation $f(x + y) = f(x) + f(y), \forall x, y \in \mathbf{R}$. Show that

(i) If f is continuous at the point $x = a$, then it is continuous for all $x \in \mathbf{R}$.

(ii) If f is continuous then $f(x) = kx$, for some constant k. **[15]**

8. (a) Using Cauchy's/Cauchy's integral formula.

(i) Evaluate $\oint_C \frac{\sin 3z}{z+\pi/2}dz$ if c is the circle $|z| = 5$.

(ii) Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}dz$ if C is : (A) the circle $|z-1| = 4$,

(B) the ellipse $|z-2|+|z+2| = 6$. **[10]**

8. (b) Find the Laurent expansion of $\frac{z}{(z+1)(z+2)}$ about the singularity $z = -2$. Specify the region of convergence. **[07]**

8. (c) By integrating $e^{iz} / (z - ai)$, $(a > 0)$ round a suitable contour, prove that

$$\int_{-\infty}^{\infty}\frac{a\cos x + x\sin x}{x^2+a^2}dx = 2\pi e^{-a}.$$ **[13]**

8. (d) Solve the following transportation problem **[20]**

	To			Available
From	2	7	4	5
	3	3	1	8
	5	4	7	7
	1	6	2	14
Required	7	9	18	34

OUR TOPPER'S MARKS LIST (IAS)

- For your final selection, optional subject marks are crucial.
- Choose Optional Subject based on Your Graduation Studies & Score Highest Marks.
- Now Mathematics has become one of the most Cherished Optional Paper among Science Graduates, especially Students with Mathematics background including B.Tech.
- In the new pattern of exam, the average marks of successful candidates in Maths is more than 300 out of 500.
- Mathematics (Opt.) has proven to be the Most Reliable and High Scoring Subject in IAS/IFoS.
- IMS has been successfully providing consistent results since its inception.

MARKS ARE BEFORE YOU AND YOU SHOULD ANALYZE YOURSELF

KANISHAK KATARIA
AIR-01
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	133
GENERAL STUDIES-I (PAPER-II)	250	098
GENERAL STUDIES-II (PAPER-III)	250	117
GENERAL STUDIES-III (PAPER-IV)	250	117
GENERAL STUDIES-IV (PAPER-V)	250	116
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	170/250	361/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	191/250	
WRITTEN TOTAL	1750	942
PERSONALITY TEST	275	179
TOTAL FINAL	2025	1121

K. VARUN REDDY
AIR-07
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	113
GENERAL STUDIES-I (PAPER-II)	250	097
GENERAL STUDIES-II (PAPER-III)	250	113
GENERAL STUDIES-III (PAPER-IV)	250	117
GENERAL STUDIES-IV (PAPER-V)	250	121
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	178/250	324/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	146/250	
WRITTEN TOTAL	1750	885
PERSONALITY TEST	275	182
TOTAL FINAL	2025	1067

TANMAY V. SHARMA
AIR-10
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	138
GENERAL STUDIES-I (PAPER-II)	250	091
GENERAL STUDIES-II (PAPER-III)	250	111
GENERAL STUDIES-III (PAPER-IV)	250	097
GENERAL STUDIES-IV (PAPER-V)	250	104
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	168/250	336/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	168/250	
WRITTEN TOTAL	1750	877
PERSONALITY TEST	275	187
TOTAL FINAL	2025	1064

G.S.S. PRAVEENCHAND
AIR-64
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	119
GENERAL STUDIES-I (PAPER-II)	250	098
GENERAL STUDIES-II (PAPER-III)	250	107
GENERAL STUDIES-III (PAPER-IV)	250	106
GENERAL STUDIES-IV (PAPER-V)	250	101
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	175/250	342/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	167/250	
WRITTEN TOTAL	1750	873
PERSONALITY TEST	275	157
TOTAL FINAL	2025	1030

MANISHA RANA
AIR-67
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	130
GENERAL STUDIES-I (PAPER-II)	250	105
GENERAL STUDIES-II (PAPER-III)	250	099
GENERAL STUDIES-III (PAPER-IV)	250	112
GENERAL STUDIES-IV (PAPER-V)	250	100
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	155/250	326/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	171/250	
WRITTEN TOTAL	1750	872
PERSONALITY TEST	275	157
TOTAL FINAL	2025	1029

DALIP KUMAR
AIR-73
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	117
GENERAL STUDIES-I (PAPER-II)	250	084
GENERAL STUDIES-II (PAPER-III)	250	115
GENERAL STUDIES-III (PAPER-IV)	250	109
GENERAL STUDIES-IV (PAPER-V)	250	097
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	171/250	327/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	156/250	
WRITTEN TOTAL	1750	849
PERSONALITY TEST	275	179
TOTAL FINAL	2025	1028

KHUSHBOO GUPTA
AIR-80
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	141
GENERAL STUDIES-I (PAPER-II)	250	088
GENERAL STUDIES-II (PAPER-III)	250	103
GENERAL STUDIES-III (PAPER-IV)	250	093
GENERAL STUDIES-IV (PAPER-V)	250	103
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	175/250	326/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	151/250	
WRITTEN TOTAL	1750	854
PERSONALITY TEST	275	171
TOTAL FINAL	2025	1025

JAY SHIVANI
AIR-81
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	117
GENERAL STUDIES-I (PAPER-II)	250	096
GENERAL STUDIES-II (PAPER-III)	250	104
GENERAL STUDIES-III (PAPER-IV)	250	098
GENERAL STUDIES-IV (PAPER-V)	250	103
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	164/250	336/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	172/250	
WRITTEN TOTAL	1750	854
PERSONALITY TEST	275	171
TOTAL FINAL	2025	1025

AANCHAL SRIVASTAVA
AIR-110
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	125
GENERAL STUDIES-I (PAPER-II)	250	090
GENERAL STUDIES-II (PAPER-III)	250	107
GENERAL STUDIES-III (PAPER-IV)	250	106
GENERAL STUDIES-IV (PAPER-V)	250	109
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	152/250	309/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	157/250	
WRITTEN TOTAL	1750	846
PERSONALITY TEST	275	171
TOTAL FINAL	2025	1017

HIMANSHU PRAJAPATI
AIR-124
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	113
GENERAL STUDIES-I (PAPER-II)	250	075
GENERAL STUDIES-II (PAPER-III)	250	104
GENERAL STUDIES-III (PAPER-IV)	250	099
GENERAL STUDIES-IV (PAPER-V)	250	094
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	168/250	328/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	160/250	
WRITTEN TOTAL	1750	813
PERSONALITY TEST	275	201
TOTAL FINAL	2025	1014

am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

It is very important for a UPSC aspirant to have genuine interest in mathematics if he/she wants to choose this optional. Maths used to be my favourite subject in school and in IITB also I had pursued additional courses in mathematics out of interest. Since the syllabus is large & requires considerable practice, it is necessary to have a genuine interest. Apart from my inherent inclination, this optional offers certain advantages which made it an obvious choice. In this optional, the marks you get are almost proportional to your efforts. With proper hard work, a candidate can comfortably attempt all the questions in exam and expect to score around 50% marks even after heavy scaling which can offer the necessary edge in this intense competition. Such candidate generally would not find any question surprising in mains. This kind of certainity is not present in humanities optionals.

THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

the best mode of judging your preparation. You can fairly evaluate your performance with your marks and then focus on the weak topics. Secondly, its a rehearsal of Mains Exam and thus helps you greatly in time management.

Mains exam is nearly a marathon for your hand and thus you get very much trained for facing them.

Test Series also provided me another pool of questions to practise. They also helped in developing the ability of answer writing which definitely can't be developed overnight. I attended Test Series of IMS and luckily many questions of Test Series appeared in both IFoS Exam and CSE. I would also request all the candidates to give the test series by coming to classroom if possible and stick to the timelines as it really helps in completion of syllabus.

I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

Thank You

Parth Jaiswal

AIR-5 in IFoS-2014,

AIR-299 in CSE-2014

KUMBHEJKAR YOGESH VIJAY
(AIR-08 in IAS-2015)

(AIR-13 IFoS) & (AIR-143 IAS)

***in* IFoS-2014 & IAS-2014 Examinations**

CLASSROOM STUDENT

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i

SUNEEL SHEORAN
AIR-192
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	118
GENERAL STUDIES-I (PAPER-II)	250	087
GENERAL STUDIES-II (PAPER-III)	250	090
GENERAL STUDIES-III (PAPER-IV)	250	105
GENERAL STUDIES-IV (PAPER-V)	250	096
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	173/250	325/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	152/250	
WRITTEN TOTAL	1750	821
PERSONALITY TEST	275	182
TOTAL FINAL	2025	1003

AKASH SINGH
AIR-193
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	114
GENERAL STUDIES-I (PAPER-II)	250	082
GENERAL STUDIES-II (PAPER-III)	250	099
GENERAL STUDIES-III (PAPER-IV)	250	095
GENERAL STUDIES-IV (PAPER-V)	250	101
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	161/250	336/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	175/250	
WRITTEN TOTAL	1750	827
PERSONALITY TEST	275	176
TOTAL FINAL	2025	1003

SACHIN BANSAL
AIR-348
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	124
GENERAL STUDIES-I (PAPER-II)	250	091
GENERAL STUDIES-II (PAPER-III)	250	109
GENERAL STUDIES-III (PAPER-IV)	250	104
GENERAL STUDIES-IV (PAPER-V)	250	105
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	167/250	316/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	149/250	
WRITTEN TOTAL	1750	849
PERSONALITY TEST	275	138
TOTAL FINAL	2025	987

KATTA RAVI TEJA
AIR-349
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	069
GENERAL STUDIES-I (PAPER-II)	250	101
GENERAL STUDIES-II (PAPER-III)	250	110
GENERAL STUDIES-III (PAPER-IV)	250	105
GENERAL STUDIES-IV (PAPER-V)	250	101
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	173/250	322/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	149/250	
WRITTEN TOTAL	1750	808
PERSONALITY TEST	275	179
TOTAL FINAL	2025	987

S.GAUTHAM RAJ
AIR-353
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	135
GENERAL STUDIES-I (PAPER-II)	250	086
GENERAL STUDIES-II (PAPER-III)	250	093
GENERAL STUDIES-III (PAPER-IV)	250	096
GENERAL STUDIES-IV (PAPER-V)	250	085
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	162/250	296/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	134/250	
WRITTEN TOTAL	1750	791
PERSONALITY TEST	275	195
TOTAL FINAL	2025	986

RAJAT BHARDWAJ
AIR-366
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	122
GENERAL STUDIES-I (PAPER-II)	250	093
GENERAL STUDIES-II (PAPER-III)	250	108
GENERAL STUDIES-III (PAPER-IV)	250	113
GENERAL STUDIES-IV (PAPER-V)	250	107
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	162/250	302/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	140/250	
WRITTEN TOTAL	1750	845
PERSONALITY TEST	275	140
TOTAL FINAL	2025	985

C. VISHNU CHARAN
AIR-406
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	105
GENERAL STUDIES-I (PAPER-II)	250	093
GENERAL STUDIES-II (PAPER-III)	250	099
GENERAL STUDIES-III (PAPER-IV)	250	090
GENERAL STUDIES-IV (PAPER-V)	250	094
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	153/250	312/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	159/250	
WRITTEN TOTAL	1750	793
PERSONALITY TEST	275	187
TOTAL FINAL	2025	980

PANKAJ KUMAWAT
AIR-443
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	093
GENERAL STUDIES-I (PAPER-II)	250	084
GENERAL STUDIES-II (PAPER-III)	250	101
GENERAL STUDIES-III (PAPER-IV)	250	115
GENERAL STUDIES-IV (PAPER-V)	250	106
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	176/250	334/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	158/250	
WRITTEN TOTAL	1750	833
PERSONALITY TEST	275	138
TOTAL FINAL	2025	971

SANJAY SAHU
AIR-526
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	111
GENERAL STUDIES-I (PAPER-II)	250	087
GENERAL STUDIES-II (PAPER-III)	250	105
GENERAL STUDIES-III (PAPER-IV)	250	106
GENERAL STUDIES-IV (PAPER-V)	250	101
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	134/250	305/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	171/250	
WRITTEN TOTAL	1750	815
PERSONALITY TEST	275	138
TOTAL FINAL	2025	953

AYUSH KUMAR
AIR-598
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	102
GENERAL STUDIES-I (PAPER-II)	250	091
GENERAL STUDIES-II (PAPER-III)	250	104
GENERAL STUDIES-III (PAPER-IV)	250	085
GENERAL STUDIES-IV (PAPER-V)	250	120
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	145/250	298/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	153/250	
WRITTEN TOTAL	1750	800
PERSONALITY TEST	275	143
TOTAL FINAL	2025	943

AMIT KUMAWAT
AIR-600
IAS-2018

SUBJECT	Max. Marks.	Marks. Obtained
ESSAY (PAPER-I)	250	118
GENERAL STUDIES-I (PAPER-II)	250	079
GENERAL STUDIES-II (PAPER-III)	250	093
GENERAL STUDIES-III (PAPER-IV)	250	103
GENERAL STUDIES-IV (PAPER-V)	250	092
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	155/250	320/500
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	165/250	
WRITTEN TOTAL	1750	805
PERSONALITY TEST	275	138
TOTAL FINAL	2025	943

REPARATION STRATEGY

for IAS/IFoS
MATHEMATICS
(Optional)
by Successful Candidate
PARTH JAISWAL
(AIR-5 IFOS) & (AIR-299 IAS)
in **IFoS-2014 & IAS-2014 Examinations**
CLASSROOM STUDENT

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

Why I Chose Mathematics?

I chose **Mathematics** because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

- This subject is vast in syllabus and takes more time to study than other optionals.
- It also requires consistent practise. But the positive part is - If you are thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.
- In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.
- So again it gives you the flexibility of giving IFoS Exam.

From where to study?

I attended classroom coaching of IMS, Rajinder Nagar. I restricted my preparation to the handouts provided by Venkanna Sir. Because of the voluminous syllabus, it is necessary to gauge the point where you have to stop. I found that the notes quite comprehensive and provided me a holistic coverage of the syllabus in a highly structured manner. I believe that those notes are sufficient from the theory point of view.

For practising questions which is of utmost importance, I solved all the questions given in the notes (whether solved or unsolved) multiple times in my registers. Besides that, I solved the questions of previous year papers provided by sir, again multiple times. I restricted my preparation upto this point. But if any student faces difficulty in understanding any particular topic or finds notes insufficient for it or wants to practise more, he/she can use any reference book for any particular topic which can easily be found on internet or available in market.

But again a word of caution, try to limit your preparation to the concepts relevant to the syllabus and don't delve into unnecessary theorems or proofs otherwise its a slippery slope to a massive ocean. We tend to skip the proofs of various theorems provided in the syllabus while studying them as they are of not much use. Proofs of theorems are generally not asked in the exams. But still I used to go through each and every proof in a brief manner provided in the notes. The reason being it would give me a better insight of the topic and often helped in me developing solutions of questions.

Test Series:

No optional is complete without writing a test series and it holds true in Maths also. Test Series is as important in your preparation as your notes + books. Firstly, Test Series is

Mains Test Series - 2019
Test - 2 (Paper - II)
Answer Key
Modern Algebra, Real Analysis, Complex Analysis & LPP

SECTION - A

Q.1.(a) Are the following groups cyclic groups? Give reasons.

(i) The Klein 4-group

(ii) The dihedral group D_4

(iii) The group of all roots (real or complex) of the equation $x^8 - 1 = 0$

(iv) The group Q^* of non-zero rationals, for multiplication.

Solution:

(i) Klein 4-group

It is not a cyclic group because order of each and every element of Klein 4-group is 2 except identity element.

Thus, no element can be a generator of entire group.

$\because$ A cyclic group of order 'n' has an element of order 'n'.

(ii) Dihedral group D_4.

It is not a cyclic group because order of D_4 is 8 whereas the order of every element is less than or equal to 4. Thus, no element can be a generator of entire group.

$\because$ A cyclic group of order 'n' has an element of order 'n'.

(iii) The group of all roots (real or complex) of the equation $x^n - 1 = 0$.

$\because x^n - 1 = 0 \Rightarrow x = 1^{1/n}$
$= e^{\frac{2\pi i}{n} l}$. $l = 0, 1, 2, \ldots, n-1$

Clearly this group is cyclic with the generator $e^{\frac{2\pi i}{n}}$.

(iv) The group Q^* of non-zero rationals, for multiplication.

$Q^* = \{ Q \setminus \{0\} \} = \{ p/q \mid p, q \neq 0, p, q \in \mathbb{R} \}$

is not a cyclic group.

Justification: Suppose it is generated by 'a', then $a^n = -1$ for some $n \in \mathbb{Z}$.

$\Rightarrow a = -1$

But $\langle -1 \rangle = \{-1, 1\} \neq Q^*$

Hence the result.

Q.1.(b) Let R be a commutative ring with unit element whose only ideals are (0) and R itself. Then R is a field.

Solution:

Let $0 \neq a \in R$,

Now, we first prove that $I = \{ax / x \in R\}$ is an ideal of R.

$\because I = \{ax / x \in R\}$

(i) Let $x = 1$

$\therefore a = a \cdot 1$

$\Rightarrow a \in I$

$\Rightarrow I \neq \phi$; Also by defn. of I, $I \subseteq R$

(ii) Now, let $v, w \in I$

$\therefore v = ax$ & $w = ay$ for some $x, y \in R$

$v - w = ax - ay$; $a, x, y \in R$

$= a(x-y)$

$= a \cdot z$ for some $z \in R$.

$\Rightarrow v - w \in I$

(iii) Let $v \in I$ (arbitrary) and $r \in R$ (arbitrary)

$v = a \cdot x$ for some $x \in R$

$v \cdot r = (a \cdot x) r = a(xr) = aw$

$\because$ ('$\cdot$' is associative)

$\therefore vw \in I$

Now,

$$r \cdot v = r \cdot (a x)$$
$$= (r \cdot a) \cdot x \quad \text{——— } \because \text{ '}\cdot\text{' is associative}$$
$$= (a \cdot r)\, x \quad \text{——— } \because \text{ '}\cdot\text{' is commutative}$$
$$= a \cdot (r \cdot x) \quad \text{——— } \because \text{ '}\cdot\text{' is associative}$$
$$= a \cdot z$$

$\therefore\ r \cdot v \in I.$

from (i), (ii) and (iii), we conclude that $I = \{ax / x \in R\}$ is an ideal of R.

Also, $I \neq \{0\}$ $\quad [\because a = a \cdot 1 \Rightarrow a \in I\ (a \neq 0)]$

But (0) and R itself are only ideals of R.
(i.e. R has no proper ideal)

$\Rightarrow I = R$

$\therefore\ 1 \in I$ ——— ($\because$ R is a ring with unity)

$\Rightarrow 1 = ax$ for some $x \in R$

[$\because$ R is commutative ring]

$\therefore\ x \cdot a = 1.$

$\Rightarrow x$ is a multiplicative inverse of a $(a \neq 0)$.

$\Rightarrow R$ is a field.

Hence, the result.

Q.1.(c) For $u_1 > 0$, the sequence u_n defined by $u_{n+1} = 1 + \frac{1}{u_n}$ $\forall n$, converges to $\left(\frac{\sqrt{5}+1}{2}\right)$

Solution:

Here, for any $u_1 > 0$,

$2 \geq u_n \geq 3/2 \quad \forall n \geq 3,$

as $2 \geq u_n \geq 3/2$,

$\Rightarrow 2 > 5/3 \geq u_{n+1} = 1 + \frac{1}{u_n} \geq 3/2 \quad \forall n \geq 3.$

So that $\forall n \geq 3$,

$$\left|u_{n+1} - u_n\right| = \frac{\left|u_n - u_{n-1}\right|}{u_n u_{n-1}} = \cdots = \frac{\left|u_4 - u_3\right|}{u_n u_{n-1}^2 \cdots u_3} \leq \frac{\left|u_4 - u_3\right|}{(3/2)^{2(n-3)}}$$

$$\Rightarrow \left|u_{n+p} - u_n\right| \leq \left|u_{n+p} - u_{n+p-1}\right| + \left|u_{n+p-1} - u_{n+p-2}\right| + \cdots + \left|u_{n+1} - u_n\right|$$

$$\leq \left|u_4 - u_3\right| \left(\frac{2}{3}\right)^{2(n-3)} \left\{1 + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^{2p}\right\}$$

$$< 2\left|u_4 - u_3\right| \left(\frac{2}{3}\right)^{2(n-3)} \longrightarrow 0 \quad \text{as } n \to \infty$$

$\Rightarrow u_n$ converges, say to l, then

$u_{n+1} = 1 + \frac{1}{u_n}$, as $n \to \infty$

$\Rightarrow l^2 - l - 1 = 0$ i.e. $l = \frac{\sqrt{5}+1}{2}$, as $l \geq \frac{3}{2}$

$\Rightarrow u_n$ converges to $\frac{\sqrt{5}+1}{2}$.

Hence, the result.

Alternatively,

Here, for any $u_1 > 0$, $2 \geq u_n \geq \frac{3}{2}$ $\forall n \geq 3$

and $\left| u_n - \frac{\sqrt{5}+1}{2} \right| = \left| u_{n-1} - \frac{\sqrt{5}+1}{2} \right| \cdot \frac{1}{u_{n-1}}$

$$\leq \left| u_3 - \frac{\sqrt{5}+1}{2} \right| \cdot \frac{1}{u_{n-1} \cdot u_{n-2} \cdots u_3}$$

$$\leq \left| u_3 - \frac{\sqrt{5}+1}{2} \right| \cdot \left(\frac{2}{3}\right)^{n-3} \to 0$$

as $n \to \infty$

$\Rightarrow u_n$ converges to $\frac{\sqrt{5}+1}{2}$.

Hence, the result.

Q.1.(d) Prove that the function $u = e^{-x}(x\cos y + y\sin y)$ is harmonic and find the corresponding analytic function.

Solution :

Let $u = e^{-x}(x\cos y + y\sin y)$

Then $$\frac{\partial u}{\partial x} = e^{-x}(\cos y) + (x\cos y + y\sin y)(-e^{-x})$$

$$= e^{-x}[\cos y - x\cos y - y\sin y]$$

$$\frac{\partial u}{\partial y} = e^{-x}(-x\sin y + y\cos y + \sin y)$$

Also, $$\frac{\partial^2 u}{\partial x^2} = e^{-x}[-\cos y] + (\cos y - x\cos y - y\sin y)(-e^{-x})$$

$$= e^{-x}[-\cos y - \cos y + x\cos y + y\sin y]$$

$$= e^{-x}[x\cos y + y\sin y - 2\cos y]$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}[-x\cos y + \cos y - y\sin y + \cos y]$$

$$= e^{-x}[-x\cos y - y\sin y + 2\cos y]$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-x}[\cancel{x\cos y} + \cancel{y\sin y} - \cancel{2\cos y} - \cancel{x\cos y} - \cancel{y\sin y} + \cancel{2\cos y}]$$

$$= 0$$

$\Rightarrow$ <u>u is harmonic.</u> —— (i)

Let v be the harmonic conjugate of u, then $f(z) = u + iv$ be analytic function.

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \left[\text{By C-R Equations, } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right]$$

$$\therefore f'(z) = e^{-x}\left[\cos y - x\cos y - y\sin y\right] - i\,e^{-x}\left[-x\sin y + y\cos y + \sin y\right] \quad \text{—— (A)}$$

Now, by applying Milne Thomson's method, replacing x by z and y by 0, in Eq^n (A), we get,

$$f'(z) = e^{-z}\left[1 - z - 0\right] - ie^{-z}[0]$$

$$f'(z) = e^{-z}\left[1 - z\right]$$

$$\Rightarrow f(z) = (1-z)(-e^{-z}) - \int -e^{-z}(-1)\,dz + \text{constant} \quad [\text{On integrating with respect to } z]$$

$$= -(1-z)e^{-z} - \int e^{-z}\,dz + \text{constant}$$

$$= -(1-z)e^{-z} - (-e^{-z}) + \text{constant}$$

$$= e^{-z}(-1 + z + 1) + \text{constant}$$

$$= e^{-x-iy}(x+iy) + \text{constant}$$

$= e^{-x}(\cos y - i\sin y)(x+iy) + \text{constant}$

$= e^{-x}(x\cos y + y\sin y) + ie^{-x}(y\cos y - x\sin y) + ic$

$f(z) = u + i[e^{-x}(y\cos y - x\sin y) + c]$ — (B)

$\therefore\ v = e^{-x}(y\cos y - x\sin y) + c$

where Eqⁿ (B) represents the required analytic function.

Hence, the result.

Q.1.(c) Write the dual of the problem.

Minimize $z = 2x_2 + 5x_3$

Subject to $x_1 + x_2 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 4$

$x_1, x_2, x_3 \geq 0.$

Solution:

First of all, we shall write the given problem in the standard primal form as follows :-

i) Since it is a minimization problem, all the constraints must involve the sign $\geq$.

$\therefore$ We multiply the second constraint by -1, and get $-2x_1 - x_2 - 6x_3 \geq -6$

(ii) The third constraint is an equality, so we replace it by two constraints.

$$x_1 - x_2 + 3x_3 \leq 4$$

and $$x_1 - x_2 + 3x_3 \geq 4$$

The first on multiplying by -1, reduce to

$$-x_1 + x_2 - 3x_3 \geq -4$$

Thus, the given problem in the standard primal form is as follows:

Minimize $Z = 0x_1 + 2x_2 + 5x_3$

subject to constraints

$$x_1 + x_2 + 0x_3 \geq 2$$
$$-2x_1 - x_2 - 6x_3 \geq -6$$
$$-x_1 + x_2 - 3x_3 \geq -4$$
$$x_1 - x_2 + 3x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

The dual of the given problem is given by

Maximize $Z^* = 2y_1 - 6y_2 - 4y_3' - 4y_3''$

subject to constraints

$$y_1 - 2y_2 - (y_3' - y_3'') \le 0$$
$$y_1 - y_2 + (y_3' - y_3'') \le 2$$
$$0 \cdot y_1 - 6y_2 - 3(y_3' - y_3'') \le 5$$
$$y_1, y_2, y_3', y_3'' \geq 0.$$

Now, writing $y_3' - y_3'' = y_3$, the dual problem is given as:

Maximize $Z^* = 2y_1 - 6y_2 + 4y_3$

subject to the contraints,

$$y_1 - 2y_2 - y_3 \le 0$$
$$y_1 - y_2 + y_3 \le 2$$
$$-6y_2 - 3y_3 \le 5$$

$y_1, y_2 \geq 0$, y_3 unrestricted in sign.

Hence, the result.

Q.2. (a)
(i) Show that every subgroup of an abelian group is normal.
(ii) Is the converse of Problem (i) true? If yes, prove it, if no, give an example of a non-abelian group all of whose subgroups are normal.

Solution :

(i) Let G be an abelian group and H be subgroup of G.

$\because$ Every subgroup of an abelian group is abelian

$\therefore$ H is an abelian group.

We have, $h \in H$ and $a \in G$

$ha = ah$

$\Rightarrow aha^{-1} \in H$

$\Rightarrow$ H is normal subgroup of G.

$\because$ By definition, Let G be a group and $H \leq G$ be a subgroup. H is said to be normal in G, denoted by $H \triangleleft G$, if for every $g \in G$ and $h \in H$ we have $ghg^{-1} \in H$, i.e. if for every $g \in G$, we have $gHg^{-1} \subseteq H$.

(ii) By the converse of Problem (i), it means that any subgroup N of a group G is normal then G must be abelian.

This is definitely not true, and it is a reason why normal subgroups are interesting and important.

a) The simplest example of a normal subgroup in a non-abelian group is the subgroup $N = \langle (1\,2\,3) \rangle \subset S_3$.

b) The smallest example is the quaternion group of order 8, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. The subgroup of order 4 is normal because it has index 2; the only subgroup of order 2 is $\{\pm 1\}$ which is normal because it equals the center of the group.

Hence, the required counter-example(s) is/are provided.

Q.2 (b) Suppose that N and M are two normal subgroups of G and that $N \cap M = \{e\}$. Show that for any $n \in N$, $m \in M$, $nm = mn$.

Solution:

Consider, $nmn^{-1}m^{-1}$ in two ways :-

Firstly,

$(nmn^{-1})m^{-1} \in M$ as $nmn^{-1} \in M$ — [M is normal subgroup of G]

and $m^{-1} \in M$. (I)

Secondly,

$n(mn^{-1}m^{-1}) \in N$ as $n \in N$ and $mn^{-1}m^{-1} \in N$

— [$\because n \in N \Rightarrow n^{-1} \in N$ and N is a normal subgroup of G] (II)

Hence, from (I) and (II),

$nmn^{-1}m^{-1} \in N \cap M = e$

$\Rightarrow nmn^{-1}m^{-1} = e$

$\Rightarrow nmn^{-1} = em$ — [Multiplying m on the right hand of each expression]

$\Rightarrow nm = emn$ — [Post-Multiplying n on both the sides]

$\Rightarrow nm = mn$ — [$e \cdot a = a$ $\forall a \in G$.]

i.e. $nm = mn$

$\forall n \in N$ and $m \in M$.

Hence, the result.

Q.2. (c)(i) Test the convergence of $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

(ii) Prove that the integral $\int_0^\infty x^{m-1}e^{-x}dx$ is convergent if and only if $m>0$.

Solution: (i)

Let $f(x) = \frac{1}{\sqrt{1-x^3}} = \frac{1}{(1-x)^{1/2}}\cdot\frac{1}{(1+x+x^2)^{1/2}}$

Clearly, $\frac{1}{(1+x+x^2)^{1/2}}$ is a bounded function and let M be its upper bound.

$\therefore f(x) \le \frac{M}{(1-x)^{1/2}}$

Also, $\int_0^1 \frac{M}{(1-x)^{1/2}}$ is convergent.

$\therefore$ By Comparison test, $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ is convergent.

Hence, the result.

(ii)

Let $f(x) = x^{m-1}e^{-x} = \frac{e^{-x}}{x^{1-m}}$

The integrand f has infinite discontinuity at 0 if $m<1$.

So, we have to examine convergence at 0 and ∞ both.

Putting $\int_0^\infty x^{m-1}e^{-x}dx = \int_0^1 x^{m-1}e^{-x}dx + \int_1^\infty x^{m-1}e^{-x}dx$,

we test the two integrals on the right for convergence at 0 and ∞ respectively.

Convergence at 0, $m<1$.

Let $g(x) = \frac{1}{x^{1-m}}$, so that $\frac{f(x)}{g(x)} = e^{-x} \to 1$ as $x \to 0$

Also, $\int_0^1 g\,dx = \int_0^1 \frac{dx}{x^{1-m}}$, converges iff $1-m<1$ i.e. $m>0$

Hence, $\int_0^1 x^{m-1}e^{-x}dx$ convergences iff $m>0$.

Convergence at ∞

Let $g(x) = \frac{1}{x^2}$, so that $\frac{f(x)}{g(x)} = \frac{x^{m+1}}{e^x} \to 0$ as $x \to \infty$, for all m.

As $\int_1^\infty \frac{dx}{x^2}$ converges, therefore $\int_1^\infty x^{m-1}e^{-x}$

also converges for all m.

Hence, $\int_0^\infty x^{m-1} e^{-x}\, dx$ is convergent iff $m>0$.

This integral, $\int_0^\infty x^{m-1} e^{-x}\, dx$, $m>0$ called Gamma function, is denoted by $\Gamma(m)$.

Hence, the result.

Q.2.(d) Apply the method of contour integration to prove that $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = \frac{\pi}{6}$.

Solution:

Let $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$ = real part of $\int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta}\, d\theta$

$= $ real part of $\int_C \frac{z^2}{5+2\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{iz}$

$\left[\begin{array}{l} \text{As, } z = e^{i\theta} \text{ then } \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) \\ \text{and } \cos 2\theta = \frac{e^{2i\theta}+e^{-2i\theta}}{2} = \frac{z^2+1/z^2}{2}. \end{array}\right]$

[Also Since, $z = e^{i\theta} \Rightarrow \frac{dz}{d\theta} = ie^{i\theta}$

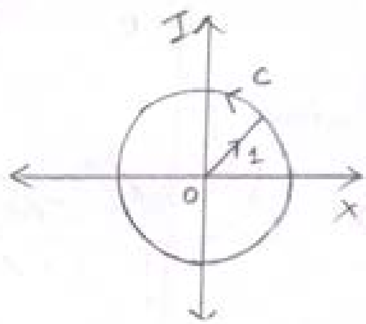
$\Rightarrow \boxed{d\theta = \frac{dz}{iz}}$]

$= \text{real part of } \frac{1}{i} \int_C \frac{z^2}{2z^2+5z+2}\, dz$

$= \text{real part of } \frac{1}{2i} \int_C \frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}\, dz$

$= \text{real part of } \int_C f(z)\, dz$ (say),

where $f(z) = \dfrac{z^2}{2i\left(z+\frac{1}{2}\right)(z+2)}$ and C is the unit circle.



Now, $z = -1/2$ and $z = -2$ are simple poles of $f(z)$

Out of those only $z = -1/2$ lies inside C.

Residue at $z = -1/2$ of order 1 is

$\lim_{z \to -1/2} \left[\left(z+1/2\right) f(z)\right] = \lim_{z \to -1/2} \cancel{\left(z+1/2\right)} \cdot \frac{z^2}{2i\cancel{\left(z+1/2\right)}(z+2)}$

$$= \lim_{z \to -1/2} \frac{z^2}{2i(z+2)}$$

$$= \frac{1}{12i}$$

$$\therefore \int_C f(z)\,dz = 2\pi i \times \text{Residue at } z = -1/2$$

$$= \cancel{2}\pi \cancel{i} \times \frac{1}{_6\cancel{12}\cancel{i}}$$

$$= \frac{\pi}{6}.$$

Hence, $\boxed{\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \text{real part of } \int_C f(z)\,dz = \frac{\pi}{6}.}$

Hence, proved.

Q.3. (a). Prove, by an example, that we can find three groups $E \subset F \subset G$, where E is normal in F. F is normal in G, but E is not normal in G.

Solution:

Let $G = S_4$.

Now, we select E and F such that

(i) $E \subset F \subset G = S_4$ and

(ii) E is normal in F,

F is normal in G but

E is not normal in G. (can be easily proved).

$\therefore$ Let E be the subgroup generated by one pair of disjoint transpositions.

i.e. $E = \langle (1\,2)(3\,4) \rangle$

and let F be the subgroup generated by all pairs of disjoint transpositions.

i.e. $F = \langle (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3) \rangle$

Hence, $\boxed{G = S_4, \ E = \langle (1\,2)(3\,4) \rangle,}$

$\boxed{F = \langle (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3) \rangle}$ is the required example.

Q.3.(b) Give an example of an integral domain which has an infinite number of elements, yet is of finite characteristic.

Solution:

Consider the ring $\mathbb{Z}_n[x]$ of polynomials in one variable x with co-efficients in $\mathbb{Z}_n$, where n is prime.

It is an infinite ring, since $x^m \in \mathbb{Z}_n[x]$ for all +ve integers m, and

$$x^{m_1} \neq x^{m_2} \text{ for } m_1 \neq m_2$$

But the characteristic of $\mathbb{Z}_n[x]$ is clearly 'n'.

Hence, the required example.

Q.3.(c) Suppose that f is defined on $I = \{x : 0 \le x \le 1\}$ by the formula

$$f(x) = \begin{cases} \frac{1}{2^n} & \text{if } x = \frac{j}{2^n} \text{ where } j \text{ is an odd integer and } 0 < j < 2^n,\ n = 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

Determine whether or not f is integrable and prove your result.

Solution:

$f(x) = \frac{1}{2^n}$ when $x = \frac{j}{2^n}$ where j is an odd integer and $0 < j < 2^n$, $n = 1, 2, \ldots$

$= \frac{1}{2}$ when $x = \frac{1}{2}$

$= \frac{1}{2^2}$ when $x = \frac{1}{2^2}, \frac{3}{2^2}$

$= \frac{1}{2^3}$ when $x = \frac{1}{2^3}, \frac{3}{2^3}, \frac{5}{2^3}, \frac{7}{2^3}$

$= \frac{1}{2^4}$ when $x = \frac{1}{2^4}, \frac{3}{2^4}, \frac{5}{2^4}, \frac{7}{2^4}, \frac{9}{2^4}, \frac{11}{2^4}, \frac{13}{2^4}, \frac{15}{2^4}$

$\vdots$

0 otherwise

$\Rightarrow f$ is bounded and continuous on $[0, 1]$ except at the points $1/2$, $1/2^2$, $3/2^2$, $1/2^3$, $3/2^3$, $5/2^3$, $7/2^3$, $1/2^4$, ...

The set of points of discontinuity of f on $[0,1]$ is $\left\{ \frac{1}{2}, \frac{1}{2^2}, \frac{3}{2^2}, \frac{1}{2^3}, \frac{3}{2^3}, \frac{5}{2^3}, \frac{7}{2^3}, \frac{1}{2^4}, \ldots \right\}$ which has only one limit point '0'.

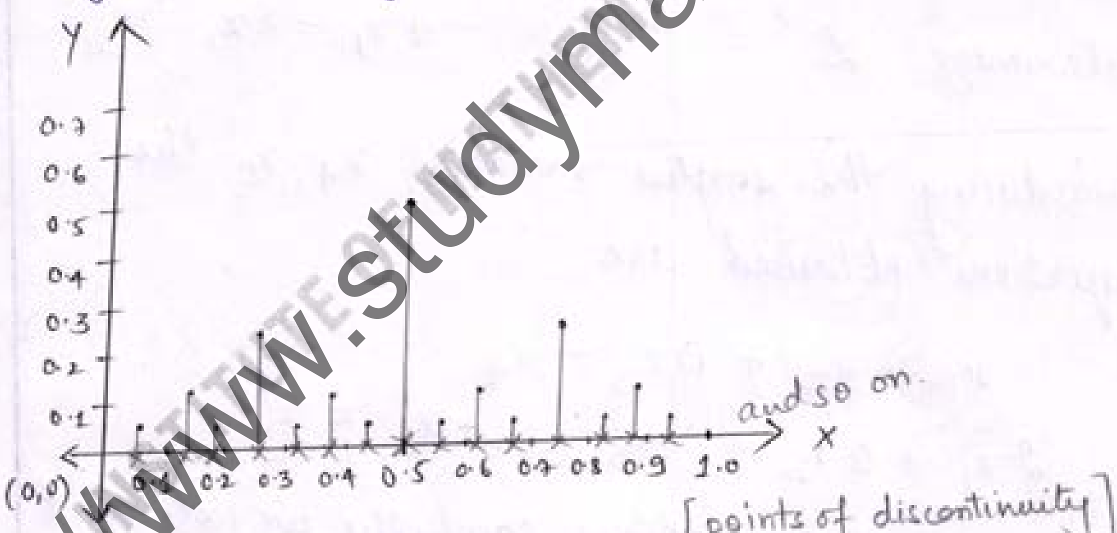
Since the set of points of discontinuity of 'f' on [0,1] has a finite number of limit points.

$\therefore$ f is integrable on [0,1].

Hence, the result.

Additional input :.

Refer to the graph of f(x):



Q.3.(d) Using simplex method, solve the L.P. problem.

Minimize $Z = 4x_1 + 8x_2 + 3x_3$

Subject to constraints

$$x_1 + x_2 \geq 2$$
$$2x_1 + x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0.$$

Solution :

First, we convert the problem of minimization to maximization problem by taking the objective function as Z'.

Maximize $Z' = -Z = -4x_1 - 8x_2 - 3x_3$

Introducing the surplus variables x_4, x_5 the equations obtained are

$$x_1 + x_2 + 0x_3 - x_4 = 2$$
$$2x_1 + 0 \cdot x_2 + x_3 - x_5 = 5$$

Examining the equations carefully we note that the columns of the coefficient of x_2 and x_3 form a unit matrix.

Therefore there is no need to introduce the artificial variables.

Taking $x_1 = 0$, $x_4 = 0$, $x_5 = 0$, we have $x_2 = 2$, $x_3 = 5$ which is the starting B.F.S.

B	C_B	c_j / X_B	Y_1 (-4)	Y_2 (-8)	Y_3 (-3)	Y_4 (0)	Y_5 (0)	Mini-Ratio X_B/Y_1
Y_2	-8	2	$\boxed{1}$	1	0	-1	0	2 (minimum) $\rightarrow$
Y_3	-3	5	2	0	1	0	-1	$5/2$
$Z' = C_B X_B = -31$		Δ_j	10 $\uparrow$	0	0	-8	-3	X_B/Y_4
Y_1	-4	2	1	1	0	-1	0	($-$ve)
Y_3	-3	1	0	-2	1	$\boxed{2}$	1	$1/2$ (minimum) $\rightarrow$
$Z' = C_B X_B = -11$		Δ_j	0	-10	0	2 $\uparrow$	-3	
Y_1	-4	$5/2$	1	0	$1/2$	0	$-1/2$	
Y_4	0	$1/2$	0	-1	$1/2$	1	$-1/2$	
$Z' = C_B X_B = -10$		Δ_j	0	-8	-1	0	-2	

Here, no $\Delta_j > 0$, so this solution is optimal.

$\therefore$ Optimal solution is

$x_1 = 5/2$, $x_2 = 0$, $x_3 = 0$ and

Minimum $Z = -Z' = 10$.

Hence, the result.

Q.4.(b) The function $f(x) = 1/x$ is continuous on $(0,1)$ but not uniformly continuous.

Solution:

Since x is continuous on $(0,1)$ and $x \neq 0$ on $(0,1)$,

$\Rightarrow \frac{1}{x}$ is continuous on $(0,1)$.

Now,
for any $\delta > 0$, $\exists$ $m \in \mathbb{N}$ such that $\frac{1}{n} < \delta$ $\forall n \geq m$.

Let $x_1 = \frac{1}{m}$, $x_2 = \frac{1}{2m}$

Then $x_1, x_2 \in (0,1)$ and $|x_1 - x_2| < \delta$

but $|f(x_1) - f(x_2)| = m$ which cannot be smaller than every $\varepsilon > 0$.

Thus, $f(x) = 1/x$ is not uniformly continuous on $(0,1)$.

Hence, the result.

Q.4.(c)
(i) Specify the nature of singularity at $z=-2$ of $f(z)=(z-3)\sin\frac{1}{z+2}$

(ii) Find zeros and poles of $\left(\frac{z+1}{z^2+1}\right)^2$.

Solution:
(i) Zeros of $f(z)$ are given by $f(z)=0$

i.e. $(z-3)\sin\frac{1}{z+2}=0 \Rightarrow z=3,\ \sin\frac{1}{z+2}=0$

Now,
$$\sin\left(\frac{1}{z+2}\right)=0=\sin n\pi$$
$$\Rightarrow \frac{1}{z+2}=n\pi$$
$$\Rightarrow z+2=\frac{1}{n\pi} \Rightarrow z=-2+\frac{1}{n\pi}\ ;\ n=0,1,2,\cdots$$

Limit point of zeros is $z=-2$.

$\therefore$ $z=-2$ is an isolated essential singularity.

(ii) We have $f(z)=\left(\frac{z+1}{z^2+1}\right)^2=\frac{(z+1)^2}{(z^2+1)^2}$

Zeros of $f(z)$ are given by $f(z)=0$
$$\Rightarrow (z+1)^2=0 \Rightarrow z=-1,-1.$$

$\therefore$ $z = -1$ is a zero of order 2.

Poles of $f(z)$ are given by.

$$(z^2+1)^2 = 0$$

$$\Rightarrow \left[(z+i)^2 (z-i)^2\right] = 0$$

$$\Rightarrow \quad z = -i, -i, i, i$$

$\therefore$ $z = i$ and $z = -i$ are poles each of order 2.

Hence, the result.

Q. 4. (d) An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of manhours that would be required for each job-man combination. This is given in the matrix form in adjacent table. Find the optimum assignment that will result in minimum manhours needed.

Man \ Job	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

Solution:

Step 1: Subtracting the smallest element of each row from all the elements of that row and then in the second matrix subtracting the smallest element of each column from all the elements of that columns the initial feasible solution determined by zeros is obtained.

Man \ Job	A	B	C	D
1	3	1	0	3
2	5	7	0	1
3	2	0	1	0
4	0	2	2	0

Table (I)

Step 2: We now examine each row successively for a single zero. Enrectangling these zeros and crossing (x) all the remaining zeros in the respective columns and then repeating same procedure for each column, we

Man \ Job	A	B	C	D
1	3	1	$\boxed{0}$	3
2	5	7	~~0~~	1
3	2	$\boxed{0}$	1	~~0~~
4	$\boxed{0}$	2	2	~~0~~

Table (II)

Step 3: We now draw the minimum number of horizontal and vertical lines necessary to cover all zeros atleast once.

Job / Man	A	B	C	D
1	3	1	$\boxed{0}$	3
2	5	7	$\cancel{0}$	1
3	2	$\boxed{0}$	1	$\cancel{0}$
4	$\boxed{0}$	2	2	$\cancel{0}$

L_2

L_3

Table (III)

Select the smallest element not covered by the lines (i.e. 1) and subtract it from all the elements not covered by the lines and add the same to the elements of the intersection of the lines.

We obtain Table (IV) providing second feasible solution to the problem.

Job / Man	A	B	C	D
1	2	0	0	2
2	4	6	0	0
3	2	0	2	0
4	0	2	3	0

Table (IV)

Step 4:

Repeating step 2, we make the 6 zero assignments as shown in Table 'V' and Table 'VI'.

Job / Man	A	B	C	D
1	2	$\boxed{0}$	$\cancel{0}$	2
2	4	6	$\boxed{0}$	$\cancel{0}$
3	2	$\cancel{0}$	2	$\boxed{0}$
4	$\boxed{0}$	2	3	$\cancel{0}$

Table (V)

Optimum Solution 1.

Man	Job	Man hours.
1.	B	3
2	C	2
3	D	7
4	A	5

17 hours

It may be noted that an assignment problem can have more than one optimum solution. The other solution is shown in Table (VI).

Job / Man	A	B	C	D
1	2	$\cancel{0}$	$\boxed{0}$	2
2	4	6	$\cancel{0}$	$\boxed{0}$
3	2	$\boxed{0}$	2	$\cancel{0}$
4	$\boxed{0}$	2	3	$\cancel{0}$

Optimum Solution-2.

Man	Job	Man hours
1	C	2
2	D	6
3	B	4
4	A	5

17 hours

Hence, the result.

SECTION – B.

Q.5. (a) Give an example of a group which is not a cyclic group, but every proper subgroup of which is cyclic.

Solution:

Let us consider Klein 4-group,

$\therefore G = \{\pm 1, \pm i, \pm j, \pm k\}$ where $i^2 = j^2 = k^2 = 1$

and $ij = k,\ jk = i,\ ki = j$.

Also, G is not a cyclic group since there is no element in G of order 4 which can generate all other elements of group G.

Klein-4-group has four proper subgroups, namely,

$H_1 = \{\pm 1\} = \langle -1 \rangle$

$H_2 = \{\pm 1, \pm i\} = \langle i \rangle$

$H_3 = \{\pm 1, \pm j\} = \langle j \rangle$

$H_4 = \{\pm 1, \pm k\} = \langle k \rangle$

Thus, every proper subgroup of G is cyclic.

Hence, Klein 4-group is the required example.

Alternatively, The group $U(8) = \{1, 3, 5, 7\}$ is not a cyclic group since $1 = 3^2 = 5^2 = 7^2 = 1$.

[i.e. no generator in $U(8)$]
The only proper subgroups are $\{1\}$, $\{1,3\}$, $\{1,5\}$ and $\{1,7\}$ which are obviously cyclic.

Q.5(b) Consider the following rings R and R' with four elements $R = \{a,b,c,d\}$ with $+$ and $\cdot$ defined by

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

·	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

and $R' = \{x,y,z,t\}$ with $+$ and $\cdot$ defined by

+	x	y	z	t
x	x	y	z	t
y	y	z	t	x
z	z	t	x	y
t	t	x	y	z

·	x	y	z	t
x	x	x	x	x
y	x	y	z	t
z	x	z	x	z
t	x	t	z	y

Investigate whether R and R' are isomorphic.

Solution: (if possible)
Let, $f : R \to R'$ be an isomorphism defined by
$$f(a) = x \quad \text{and} \quad f(d) = y$$
as a and x are additive identities of R and R'

respectively, whereas α and y are multiplicative identities of R and R'.

$\because$ f is one-one, we have following two cases for $f(b)$ and $f(c)$.

Case i:

$f(b) = z,\ f(c) = t$

$\therefore f(b\cdot c) = f(a) = x$

$\therefore f(b)\cdot f(c) = z\cdot t = z$

Thus, $f(b\cdot c) \neq f(b)\cdot f(c)$

$\Rightarrow f$ is not a homomorphism

$\Rightarrow f$ is not an isomorphism.

Case ii:

$f(b) = t,\ f(c) = z$

$\therefore f(b\cdot c) = f(a) = x$

$\therefore f(b)\cdot f(c) = t\cdot z = z$

Again, $f(b\cdot c) \neq f(b)\cdot f(c)$

$\Rightarrow f$ is not a homomorphism

$\Rightarrow f$ is not an isomorphism.

Thus, R and R' are not isomorphic to each other.

Hence, the result.

Q.5.(c) The sequence $nx/(1+n^2x^2)$ is not uniformly convergent over $\mathbb{R}$ but it is uniformly convergent on $\{x : |x| > k > 0\}$.

Solution:

We know that $nx/(1+n^2x^2)$ converges to '0' for any $x \in \mathbb{R}$.

Let the convergence be uniform on $\mathbb{R}$ then for $\epsilon = \frac{1}{20}$ $\exists$ $m \in \mathbb{N}$ such that

$$\left| \frac{2nx}{1-4n^2x^2} - \frac{nx}{1+n^2x^2} \right| < \frac{1}{20} \quad \forall\, n \geq m$$

and $x \in \mathbb{R}$.

(For $p = n$ by Cauchy's principle).

On putting $x = 1/m$, $n = m$, we get,

$$\left| \frac{2}{5} - \frac{1}{2} \right| = \frac{1}{10} < \frac{1}{20}$$

which is a contradiction.

This contradiction arises due to our wrong assumption.

Therefore, $nx/(1+n^2x^2)$ is not uniformly convergent on $\mathbb{R}$.

Further, on $\{x : |x| > k > 0\}$ for $\varepsilon > 0$

$$\left|\frac{nx}{1+n^2x^2} - 0\right| < \frac{1}{|nx|} < \frac{1}{nx} < \varepsilon$$

$$\forall\ n \geq \left[\frac{1}{k\varepsilon}\right]$$

Hence, $\frac{nx}{(1+n^2x^2)}$ converges uniformly on $\{x : |x| > k > 0\}$.

Hence, the result.

Q.5.(d) If $f(z) = \frac{x^3y(y-ix)}{x^6+y^2}$, $z \neq 0$ and $f(0) = 0$, show that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

Solution:

We have, $\frac{f(z)-f(0)}{z} = \frac{f(z)-0}{z} = \frac{f(z)}{z}$

$$= \frac{x^3y\,(y-ix)}{(x^6+y^2)(x+iy)}$$

$$= \frac{-i x^3 y (\cancel{x+iy})}{(x^6+y^2)(\cancel{x+iy})}$$

$$= \frac{-i x^3 y}{x^6+y^2}$$

Let $z \to 0$ along the radius vector $y = mx$.

Then we have,

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{-i x^3.(mx)}{x^6+(mx)^2}$$

$$= \lim_{x\to 0} \frac{-i m x^2}{x^4+m^2}$$

$$= 0$$

i.e. $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector. — (i).

Now, let $z \to 0$ along $y = x^3$, then

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{-i x^3(x^3)}{x^6+(x^3)^2} = \frac{-i}{2} \neq 0$$

i.e. $\frac{f(z)-f(0)}{z} \not\to 0$ as $z \to 0$ along any other manner. Hence, the result.

Q.5. (e) A firm plans to purchase atleast 200 quintals of scrap containing high quality metal X and low quality metal Y. It decides that the scrap to be purchased must contain atleast 100 quintals of X-metal and not more than 30 quintals of Y-metal. The firm can purchase the scrap from two suppliers (A and B) in unlimited quantities. The percentage of X and Y metals in terms of weight in the scraps supplied by A and B is given below:

Metals	Supplier A	Supplier B
X	25%	75%
Y	10%	20%

The price of A's scrap is ₹200 per quintal and that of B's scrap is ₹400 per quintal. Formulate this problem as LP model and solve it graphically to determine the quantities that the firm should buy from the two suppliers, so as to minimize total purchase cost.

Solution:

The formulation of the given problem is:

Minimize (total cost) $Z = 200x_1 + 400x_2$

subject to the constraints:

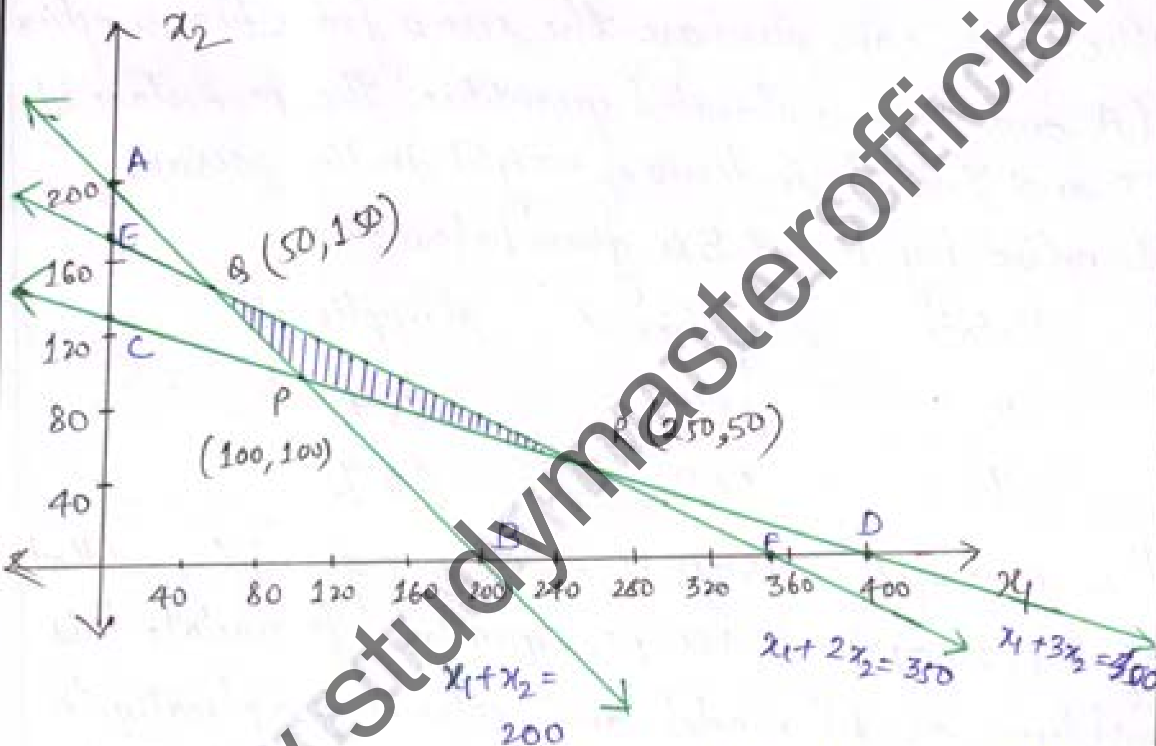
$$x_1 + x_2 \geq 200,$$

$$\tfrac{1}{4}x_1 + \tfrac{3}{4}x_2 \geq 100,$$

$$\frac{1}{10} x_1 + \frac{1}{5} x_2 \leq 35,$$

$$x_1, x_2 \geq 0$$

where x_1, x_2 represent the number of quintals of scrap from two suppliers A and B respectively.



The feasible region is the shaded area PQR which is obtained by drawing the graph of the given constraints.

The co-ordinates of the corner points of the feasible region are $P(100, 100)$, $Q(50, 150)$, $R(250, 50)$.

Thus, Z has minimum value at the point $P(100, 100)$.

Hence, the required answer is $\boxed{x_1 = 100, x_2 = 100}$

$\boxed{\text{minimum } Z = ₹\ 60,000}$

Q.6.(a)

(i) Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo p, p is a prime number, such that $ad-bc \neq 0$. G forms a group relative to matrix multiplication. What is $o(G)$?

(ii) Let H be the subgroup of G of part (i) defined by $H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \;/\; ad-bc = 1 \right\}$.

What is $o(H)$?

Solution:

(i) For the first row (a, b) of a matrix in G, a and b could be anything in $\mathbb{Z}_p$, but we must exclude the case $a=0$ and $b=0$. $[\because ad-bc \neq 0]$

Hence, $(p\times p)-1$ possibilities for the first row.

Now, for the second row, we note that it should NOT be a multiple of the first row.

Hence, for second row, we have $(p\times p)-p$ possibilities.

$\Rightarrow$ The number of elements in G is

$(p^2-1)\cdot(p^2-p)$.

i.e. $\boxed{o(G) = (p^2-1)(p^2-p)}$ ——— (i)

(ii)

The set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad-bc=1$

forms a subgroup H of G.

Moreover for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ either

$ad-bc = \det(g) = 1$ or $\det(g) = -1$

$[\equiv (p-1) \pmod p]$

On the other hand, for any,

$Hg_1, Hg_2 \in G$, $Hg_1 = Hg_2$ iff

$\det(g_1) = \det(g_2)$.

Hence, the above subgroup has index 2 in G.

i.e. $\boxed{o(H) = \frac{(p^2-1)(p^2-p)}{2}}$ ——— (ii)

Q.6.(b) Investigate whether the following ideals are (i) prime ideals, (ii) maximal ideals in the ring R.

(i) $R = \mathbb{Z}_6$ and $I = \{\bar{0}, \bar{2}, \bar{4}\}$.

(ii) $R = 2\mathbb{Z}$ and $I = 4\mathbb{Z}$

(iii) $R = C[0,1]$ the ring of continuous functions on $[0,1]$ and $I = \{x \mid x(1/2) = 0\}$.

Solution:

(i) I is a maximal ideal of R as
$R/I \simeq \mathbb{Z}$ (field).

Hence, it also a prime ideal.

(ii) I is a maximal ideal of R

If $J = n\mathbb{Z} \supset I$, 2 divides n which divides 4.

Hence, $n = 2$ or $n = 4$.

i.e. $J = R$ or $J = I$.

It is also a prime ideal.

(iii) I is a prime ideal of R.

Consider the map $\theta : C[0,1] \to F$ [F reals]

defined by $\theta(x) = x(1/2)$

It is a ring homomorphism with

$\text{Ker}\,\theta = I$.

Hence, $\dfrac{C[0,1]}{I} \simeq \text{Im}(\theta)$

Also, θ is 'onto' because given any value λ of F, there exists a continuous map x on $[0,1]$ with $x(1/2) = \lambda$.

Since F is a field, I is a maximal ideal and also a prime ideal.

Q.6.(c) Prove that every integral domain can be imbedded in a field.

Solution:

Let D be an integral domain with atleast two elements.

Let us consider $S = \{(a,b) / a, b \in D : b \neq 0\}$ then $S \neq \phi$ and $S \subset D \times D$.

$\forall (a,b), (c,d) \in S$

Define a relation '$\sim$' on 'S' as

$(a,b) \sim (c,d) \Leftrightarrow ad = bc$

We now prove that $\sim$ is an equivalence relation on S.

(i) For each $(a,b) \in S$

We have $ab = ba$

$\Rightarrow (a,b) \sim (a,b)$

(ii) for $(a,b), (c,d) \in S$

We have $(a,b) \sim (c,d)$

$\Rightarrow ad = bc$

$\Rightarrow cb = da$

$\Rightarrow (c,d) \sim (a,b)$

(iii) for $(a,b), (c,d), (e,f) \in S$

$(a,b) \sim (c,d)$, $(c,d) \sim (e,f)$

$\Rightarrow ad = bc$, $cf = de$

$\Rightarrow (ad)f = (bc)f$, $cf = de$

$\Rightarrow (af)d = b(de)$

$\Rightarrow (af)d = b(ed)$

$\Rightarrow (af)d = (be)d$

$\Rightarrow af = be \quad (\because d \neq 0)$

$\Rightarrow (a,b) \sim (e,f)$

$\therefore$ '$\sim$' is an equivalence relation on 'S'.

The equivalence relation '$\sim$' partitions the set 'S' into equivalence classes which are identical or disjoint.

For $(a,b) \in S$,

let $\frac{a}{b}$ denote equivalence class of (a,b). then

$\frac{a}{b} = \{(x,y) \in S \,/\, (x,y) \sim (a,b)\}$

i.e. $\frac{a}{b} = \{(x,y) \in S \,/\, (x,y) \sim (a,b) \Leftrightarrow xb = ya\}$

If $\frac{a}{b}, \frac{c}{d}$ are the equivalence classes of (a,b), $(c,d) \in S$, then either $\frac{a}{b} = \frac{c}{d}$ or $\frac{a}{b} \cap \frac{c}{d} = \phi$

It is evident that $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$.

Let F denote the set of all the equivalence classes or the set of quotients then

$$F = \left\{ \frac{a}{b} \,/\, (a,b) \in S \right\}$$

Since D has atleast two elements say $0, a \in D$

We have quotients $\frac{0}{a}, \frac{a}{a} \in F$ and $\frac{0}{a} \neq \frac{a}{a}$.

$\therefore$ the set F has atleast two elements.

For $\frac{a}{b}, \frac{c}{d} \in F$, define addition $(+)$ and multiplication $(\cdot)$

as

(i) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and

(ii) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

Since D is without zero divisors,

$$\underbrace{0 \neq b, d \neq 0}_{\in D} \Rightarrow bd \neq 0.$$

so $\frac{ad+bc}{bd}, \frac{ac}{bd} \in F$.

Now, we prove that the $+^n$ and $\times^n$ defined above are well-defined.

Let $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$

then $ab' = a'b$ and $cd' = c'd$ — (I)

Now, (I) $\Rightarrow$ $ab'dd' = a'bdd'$ and

$bb'cd' = bb'c'd$

$\Rightarrow ab'dd' + bb'cd' = a'bdd' + bb'c'd$

$\Rightarrow (ad+bc)b'd' = (a'd'+b'c')bd$

$\Rightarrow \dfrac{ad+bc}{bd} = \dfrac{a'd'+b'c'}{b'd'}$

Also (I) $\Rightarrow ab'cd' = a'bc'd$

$\Rightarrow (ac)(b'd') = (a'c')(bd)$

$\Rightarrow \dfrac{ac}{bd} = \dfrac{a'c'}{b'd'}$

$\therefore$ $+^n$ and $\times^n$ of quotients are well-defined binary operations on F.

We now prove that $(F, +, \cdot)$ is a field:

(i) for $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$;

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f} = \frac{ad+bc}{bd}+\frac{e}{f}$$

$$= \frac{(ad+bc)f+(bd)e}{(bd)f}$$

$$= \frac{a(df)+(cf+de)b}{b(df)}$$

$$= \frac{a}{b}+\frac{cf+de}{df}$$

$$= \frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$$

$\therefore$ addition is associative.

(ii) for $\frac{a}{b}, \frac{c}{d} \in F$; $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

$$= \frac{bc+ad}{db}$$

$$= \frac{c}{d} + \frac{a}{b}$$

$\therefore$ addition is commutative.

(iii) for $u \neq 0 \in D$ we have $\frac{0}{u} \in F$ such that

$$\frac{0}{u} + \frac{a}{b} = \frac{0b+ua}{ub} = \frac{ua}{ub} = \frac{a}{b} \quad \forall \frac{a}{b} \in F$$

$\therefore$ $\frac{0}{u} \in F$ is the zero element.

(iv) Let $\frac{a}{b} \in F$ then $-\frac{a}{b} \in F$ such that

$$\frac{a}{b} + \left(-\frac{a}{b}\right) = \frac{ab+(-a)b}{b^2} = \frac{0}{b^2} = \frac{0}{u}$$

$(\because 0u = 0b^2)$

$\therefore$ every element in F has additive inverse.

(v) for $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$;

$$\left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{ac}{bd} \cdot \frac{e}{f}$$

$$= \frac{(ac)e}{(bd)f}$$

$$= \frac{a(ce)}{b(df)} = \frac{a}{b} \cdot \frac{ce}{df}$$

$$= \frac{a}{b}\cdot\left(\frac{c}{d}\cdot\frac{e}{f}\right)$$

$\therefore$ multiplication is associative.

(vi) for $\frac{a}{b}, \frac{c}{d} \in F$;

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd} = \frac{ca}{db} = \frac{c}{d}\cdot\frac{a}{b}$$

$\therefore$ multiplication is commutative.

(vii) for $u \neq 0 \in D$, we have $\frac{u}{u} \in F$ such that $\frac{a}{b}\cdot\frac{u}{u} = \frac{au}{bu} = \frac{a}{b}$ $\forall$ $\frac{a}{b} \in F$.

$\therefore$ $\frac{u}{u} \in F$ is the unity element.

(viii) Let $\frac{a}{b} \in F$ and $\frac{a}{b} \neq \frac{0}{u}$.

Then $au \neq 0 \Rightarrow a \neq 0$ and already $u \neq 0$

$\therefore$ $b \neq 0$ and $a \neq 0 \Rightarrow \frac{b}{a} \in F$

$\therefore$ for $\frac{a}{b}\left(\neq \frac{0}{u}\right) \in F$ there exists $\frac{b}{a} \in F$ such that $\frac{a}{b}\cdot\frac{b}{a} = \frac{ab}{ba} = \frac{u}{u}$ $\left[\because (ab)u = (ba)u\right]$

$\therefore$ Every non-zero element in F has multiplicative inverse.

(ix) for $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$;

$$\frac{a}{b}\cdot\left(\frac{c}{d}+\frac{e}{f}\right) = \frac{a}{b}\cdot\frac{cf+de}{df}$$

$$= \frac{a(cf+de)}{b(df)}$$

$$= \frac{(acf+ade)(bdf)}{(bdf)(bdf)} \quad \left[\because \frac{bdf}{bdf} = \frac{u}{u}\right]$$

$$= \frac{acf\,bdf + ade\,bdf}{(bdf)(bdf)}$$

$$= \frac{acf}{bdf} + \frac{ade}{bdf}$$

$$= \frac{ac}{bd} + \frac{ae}{bf}$$

$$= \frac{a}{b}\cdot\frac{c}{d} + \frac{a}{b}\cdot\frac{e}{f}$$

Similarly, we can prove that,

$$\left(\frac{c}{d}+\frac{e}{f}\right)\cdot\frac{a}{b} = \frac{c}{d}\cdot\frac{a}{b} + \frac{e}{f}\cdot\frac{a}{b}$$

$\therefore$ multiplication is distributive over addition.

Thus, in view of (i) to (ix) results,

$(F, +, \cdot)$ is a field.

Now, we have to prove that D is embedded in the field F, i.e. we have to show that there exists an isomorphism of D into F.

Define the mapping $\phi : D \to F$ by
$\phi(a) = \frac{ax}{x}$ $\forall\, a \in D$ and $x(\neq 0) \in D$.

$a, b \in D$ and $\phi(a) = \phi(b)$

$\Rightarrow \frac{ax}{x} = \frac{bx}{x}$

$\Rightarrow (ax)x = (bx)x$

$\Rightarrow (a-b)x^2 = 0$

$\Rightarrow a-b = 0$ since $x^2 \neq 0$.

$\Rightarrow a = b$.

$\therefore$ ϕ is one-one.

for $a, b \in D$;

$\phi(a+b) = \frac{(a+b)x}{x} = \frac{(a+b)xx}{xx} = \frac{axx+bxx}{xx}$

$= \frac{axx}{xx} + \frac{bxx}{xx} = \frac{ax}{x} + \frac{bx}{x} = \phi(a) + \phi(b)$ —— (i)

$\phi(ab) = \frac{(ab)x}{x} = \frac{(ab)xx}{xx} = \frac{ax}{x} \cdot \frac{bx}{x}$

$= \phi(a)\cdot\phi(b)$ ——— (ii)

from (i) & (ii),

ϕ is a homomorphism.

Hence, ϕ is an isomorphism of D into F.

$\therefore$ The Integral Domain D is embedded in the field F.

Hence, the theorem.

Q.7.(a) Prove that the following sets are bounded.

$\{n^{1/n} : n \in \mathbb{N}\}$, $\left\{\left(1+\frac{1}{n}\right)^n : n \in \mathbb{N}\right\}$, $\{a^{1/n} : a > 0$ and $n \in \mathbb{N}\}$. Give supremum and infimum of each of these sets.

Solution:

(i) $\{n^{1/n} : n \in \mathbb{N}\} = \{1, 2^{1/2}, 3^{1/3}, 4^{1/4}, \ldots\}$

as, we know that $\lim_{n\to\infty} n^{1/n} = 1$.

$\Rightarrow$ The set is bounded. [$\because$ Every cgt seq. is bounded.]

Here, the infimum is 1 and the supremum is $3^{1/3}$.

(ii) $\left\{\left(1+\frac{1}{n}\right)^n : n \in \mathbb{N}\right\}$

Also, we know that $\lim\limits_{n\to\infty} \left\{\left(1+\frac{1}{n}\right)^n : n\in\mathbb{N}\right\} = e$

$\Rightarrow$ It is bounded.

Here, the infimum is 2 and supremum is e.

Additional steps to show limit :-

$$e^{\ln(1+1/n)^n} = e^{n\ln(1+1/n)}$$

$$\lim_{n\to\infty}(1+1/n)^n = \lim_{n\to\infty} e^{n\ln(1+1/n)}$$

$$= \lim_{n\to\infty} n\ln(1+1/n)$$

$$= e$$

$$= e^{\lim\limits_{n\to\infty} \frac{\ln(1+1/n)}{1/n}}$$

Applying L'Hopital's rule,

$$= e^{\lim\limits_{n\to\infty} \frac{\frac{1}{1+1/n}\cdot\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{\left(-1/n^2\right)}}}$$

$\therefore \lim\limits_{n\to\infty}(1+1/n)^n = e$

Similarly, $\lim\limits_{n\to\infty} n^{1/n} = 1$ can be proved.

(iii) $\{a^{1/n} : a > 0 \text{ and } n \in \mathbb{N}\}$

Let $L = \lim_{n\to\infty} a^{1/n}$

Taking log on both sides, we have

$$\log L = \lim_{n\to\infty} \frac{1}{n} \log a$$

$$\Rightarrow \underline{L = 1}.$$

i.e. for any $a > 0$, $(a^{1/n})$ converges to 1.
$\Rightarrow$ It is bounded.

Here infimum is and supremum is .

Hence, the result.

Q.7.(b) Show that $\prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right)$ converges and its limit lies between $1/2$ and 1.

Solution:
The given infinite product is $\prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right)$

$$= \prod_{n=1}^{\infty}(1 - b_n), \text{ where } b_n = \frac{1}{4n^2}$$

and $n \geq 1$

so that $0 < b_n < 1$.

$\therefore$ The product $\prod_{n=1}^{\infty}(1-b_n)$ and the series $\sum_{n=1}^{\infty} b_n$ converge or diverge together.

But the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{4n^2}$ is convergent

[By P-Test]

$\therefore$ The given product $\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right)$ is convergent.

Now,

$\because \; 0 < b_n < 1$

$\Rightarrow -1 < -b_n < 0$

$\Rightarrow 1-1 < 1-b_n < 1$

$\Rightarrow \quad 0 < 1-b_n < 1.$

$\Rightarrow \prod_{n=1}^{\infty}(0) < \prod_{n=1}^{\infty}(1-b_n) < \prod_{n=1}^{\infty}(1)$

$$\Rightarrow \quad 0 < \prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right) < 1 \quad \text{———} \quad (*)$$

Also, $1-\frac{1}{4n^2}$ is an increasing function with increasing n.

$$\left[\because \text{ Let } f(x) = 1 - \frac{1}{4n^2},\ f'(x) = \frac{1}{2n^3} > 0 \ \forall\, n > 0\right]$$

$$\Rightarrow \prod_{n=1}^{\infty}(1 - b_n) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right) = \prod_{n=1}^{\infty}\left(\frac{4n^2-1}{4n^2}\right)$$

$$= \prod_{n=1}^{\infty}\left(\frac{2n+1}{2n}\right)\left(\frac{2n-1}{2n}\right) = \prod_{n=1}^{\infty}\left(1 + \frac{1}{2n}\right)\left(1 - \frac{1}{2n}\right)$$

$$> \left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)$$

$$> 1 \cdot \frac{1}{2}$$

$$= \frac{1}{2}$$

i.e. $\displaystyle \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right) > \frac{1}{2}$ —— (**)

From (*) and (**), we conclude that, the limit of $\displaystyle \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right)$ lies between $\frac{1}{2}$ and 1.

Hence, the result.

Q.7.(c) for $x > -1$, test for convergence

$$1 + \frac{2^2}{3.4}x + \frac{2^2.4^2}{3.4.5.6}x^2 + \frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^3 + \cdots$$

Solution:

Leaving the first term,

$$u_n = \frac{2^2.4^2.\cdots\cdots.(2n)^2}{3.4.5.6.\cdots.(2n+2)} x^n$$

$$u_{n+1} = \frac{2^2.4^2.\cdots\cdots.(2n)^2(2n+2)^2}{3.4.5.6\cdots\cdots.(2n+2)(2n+3)(2n+4)}.x^{n+1}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{(2n+4)(2n+3)}{(2n+2)^2}.\frac{1}{x}$$

$$= \frac{(2+4/n)(2+3/n)}{(2+2/n)^2}.\frac{1}{x}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio test,

Σu_n converges if $1/x > 1$ i.e. $x < 1$

and Σu_n diverges if $1/x < 1$ i.e. $x > 1$.

If $x=1$, then the ratio test fails.

when $x=1$,

$$\frac{u_n}{u_{n+1}} = \frac{(2n+4)(2n+3)}{(2n+2)^2}$$

$$= \frac{(1+4/2n)(1+3/2n)}{(1+2/2n)^2}$$

$$= \frac{\left[1+7/2n+\cancel{12}^{3}/\cancel{4}n^2\right]}{\left[1+\frac{2\cdot\cancel{2}}{\cancel{2}n}+\frac{\cancel{4}}{\cancel{4}n^2}\right]}$$

$$= \left[1+\frac{7}{2n}+\frac{3}{n^2}\right]\left[1+\left(\frac{2}{n}+\frac{1}{n^2}\right)\right]^{-1}$$

$$= \left(1+\frac{7}{2n}+\frac{3}{n^2}\right)\left(1-\frac{2}{n}-\frac{1}{n^2}+\frac{4}{n^2}+\frac{1}{n^4}-\frac{8}{n^3}-\frac{1}{n^6}+\cdots\right)$$

$$= \left(1+\frac{7}{2n}+\frac{3}{n^2}\right)\left(1-\frac{2}{n}+\frac{3}{n^2}-\frac{8}{n^3}+\frac{1}{n^4}+\cdots\right)$$

$$= \left(1-\frac{2}{n}+\frac{3}{n^2}-\frac{8}{n^3}+\cdots\right)+\left(\frac{7}{2n}-\frac{14}{2n^2}+\frac{21}{2n^3}-\cdots\right)+\left(\frac{3}{n^2}-\frac{6}{n^3}+\frac{9}{n^4}-\cdots\right)$$

$$= 1+\frac{3}{2n}-\frac{1}{n^2}+\cdots$$

$$= 1 + \left(\frac{3}{2}\right)\left(\frac{1}{n}\right) + O\left(\frac{1}{n^2}\right)$$

Comparing it with

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

where $\lambda = \frac{3}{2} > 1$.

$\therefore$ By Gauss Test,

Σu_n is convergent at $x=1$.

$\therefore$ The given series converges if $x \leq 1$ and diverges if $x > 1$.

Hence, the result.

Q.7.(d) Let a function $f: \mathbb{R} \to \mathbb{R}$ satisfy the equation $f(x+y) = f(x) + f(y)$, $\forall\, x, y \in \mathbb{R}$.

Show that

(i) If f is continuous at the point $x = a$, then it is continuous for all $x \in \mathbb{R}$.

(ii) If f is continuous then $f(x) = kx$, for some constant k.

Solution:

(ii)

Let f be continuous at the point $x=a$.

We have $f(a+0) = \lim_{h\to 0} f(a+h)$

$$= \lim_{h\to 0} [f(a)+f(h)]$$

— (by defn. of f)

$$= \lim_{h\to 0} f(a) + \lim_{h\to 0} f(h)$$

$$= f(a) + \lim_{h\to 0} f(h)$$

$\because$ f is continuous at a, we have,

$$f(a) = f(a+0)$$

$$\Rightarrow f(a) = f(a) + \lim_{h\to 0} f(h)$$

$$\Rightarrow \lim_{h\to 0} f(h) = 0 \quad \text{—— (1)}$$

Similarly, $f(a) = f(a-0) = \lim_{h\to 0} f(-h) = 0$ —— (2)

Now, let α be any real number.

To show that: f is continuous at α.

We have, $f(\alpha+0) = \lim_{h\to 0} f(\alpha+h)$

$$= \lim_{h\to 0} [f(\alpha) + f(h)]$$

— (defn. of f)

$$= \lim_{h\to 0} f(\alpha) + \lim_{h\to 0} f(h)$$

$$= f(\alpha) \quad \text{--- using (1)}$$

Similarly,
using (2), we have $f(\alpha-0) = f(\alpha)$

Thus, $f(\alpha+0) = f(\alpha-0) = f(\alpha)$.

Hence, f is continuous at $x=\alpha$.
Since α is arbitrary, so f is continuous for all $x \in \mathbb{R}$. —— (i)
Hence, proved

(ii)
Here, we consider the following cases:-

<u>Case (I)</u>: Let $x=0$

$\because f(x+x) = f(x)+f(x)$, we have
$f(0) = f(0+0) = f(0)+f(0)$ and so
$f(0) = 0$.

Hence, $f(x) = kx$ for every constant k in this case.

<u>Case (II)</u>: Let x be any +ve integer, then

$f(x) = f(1+1+\cdots\ x\text{ times}) = \underbrace{f(1)+f(1)+\cdots}_{x\text{ times}}$

$= x\cdot f(1) = k\cdot x$, where $k = f(1)$ i.e. constant

Case (III): Let x be any $-$ve integer, then Put $x = -y$ so that y is a +ve integer.

Now $0 = f(0)$ — [by case (I)]

$= f(y-y)$

$= f(y) + f(-y)$ (by defn of f)

$\therefore f(-y) = -f(y)$

Hence, $f(x) = -f(y) = -y\cdot f(1)$ — (by Case II)

$= k\cdot x$ where $k = -f(1)$ i.e. constant

Case (IV): Let x be any rational number.

Put $x = p/q$ where q is a +ve integer and p is any integer, +ve, $-$ve or zero.

Now, $f(q\cdot p/q) = f(\underbrace{p/q + p/q + \cdots}_{q \text{ times}})$

$= \underbrace{f(p/q) + f(p/q) + \cdots}_{q \text{ times}}$

$\therefore f(p) = q\cdot f(p/q)$

But $f(p) = kp$ by previous cases.

Hence, $q \cdot f(p/q) = kp$

$\Rightarrow f(p/q) = k \cdot p/q$

$\Rightarrow f(x) = k \cdot x$, in this case also.

Case (V): finally,

Let x be any real number.

Let $\langle x_n \rangle$ be a sequence of rational numbers converging to x.

Since, f is continuous at x, the sequence $\langle f(x_n) \rangle$ converges to $f(x)$.

Thus, we have

$\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} f(x_n) = f(x)$

$\because$ x_n is a rational number, we have by case (IV), $f(x_n) = k \cdot x_n$

$\therefore \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} k \cdot x_n = k \cdot \lim_{n\to\infty} x_n = k \cdot x$.

$\Rightarrow f(x) = k \cdot x$.

Hence, $f(x) = kx \;\forall\; x \in \mathbb{R}$ —— (ii)

Hence, proved.

Q.8.(b) Find the Laurent expansion of $\frac{z}{(z+1)(z+2)}$ about the singularity $z=-2$.

Specify the region of convergence.

Solution:

We have $f(z) = \frac{z}{(z+1)(z+2)} = \frac{2}{z+2} - \frac{1}{z+1}$

$$\Rightarrow f(z) = \frac{2}{z+2} - \frac{1}{z+1} \quad \text{—— (1)}$$

To find Laurent expansion for $\phi(z) = \frac{1}{z+1}$ about $z=-2$, we write

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z+2)^n \quad \text{—— (2)}$$

where $a_n = \frac{\phi^{(n)}(-2)}{n!}$.

But $\phi^{(n)}(z) = \frac{(-1)^n n!}{(z+1)^{n+1}}$

$$\therefore \frac{\phi^{(n)}(-2)}{n!} = \frac{(-1)^n}{(-2+1)^{n+1}} = \frac{(-1)^n}{(-1)^{n+1}} = -1.$$

or $a_n = \frac{\phi^{(n)}(-2)}{n!} = -1.$ —— (3)

Put ③ in ②, we get,

$$\frac{1}{z+1} = \phi(z) = \sum_{n=0}^{\infty} (-1)(z+2)^n$$

$$\Rightarrow -\frac{1}{z+1} = \sum_{n=0}^{\infty} (z+2)^n$$

Now, ① reduces to $\boxed{f(z) = \frac{2}{2+z} + \sum_{n=0}^{\infty} (z+2)^n}$ —— ④

which is the required expansion.

Let $\sum_{n=0}^{\infty} (z+2)^n = \sum u_n$

Then $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(z+2)^{n+1}}{(z+2)^n}\right| = |z+2|$

series will be convergent if $\left|\frac{u_{n+1}}{u_n}\right| < 1$

i.e. if $|z+2| < 1$

$\therefore$ Radius of convergence $= 1$.

Series is convergent $\forall z$, inside the circle whose centre is $z = -2$ and radius $= 1$ unit.

Hence, the result.

Q.8.(c). By integrating $e^{iz}/(z-ai)$, $(a>0)$ round a suitable contour, prove that

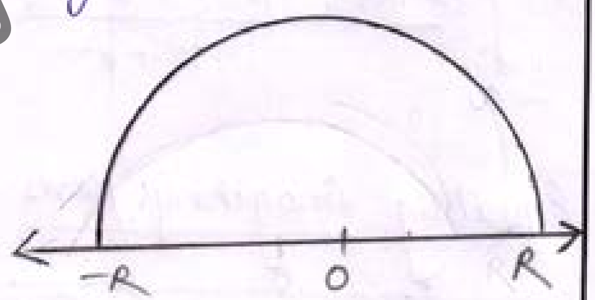
$$\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2}\,dx = 2\pi e^{-a}.$$

Solution:

Consider the integral, $\int_C \frac{e^{iz}}{(z-ai)}\,dz = \int_C f(z)\,dz$,

where C is the contour consisting of a large semi-circle Γ of radius R containing all the poles of the integrand in the upper half plane and the part of real axis from $-R$ to R.

$z = ia$ is the simple pole of $f(z)$ and it lies inside C.



$\therefore$ Residue at $z = ia = \lim_{z\to ia} (z-ia) f(z)$

$$= \lim_{z\to ia} e^{iz} = \boxed{e^{-a}}$$

By Cauchy's Residue theorem, we have,

$$\int_C f(z)\,dz = \int_{-R}^{R} \frac{e^{iz}}{z-ia}\,dz + \int_{\Gamma} \frac{e^{iz}}{z-ia}\,dz = 2\pi i \Sigma R$$

By Jordan's lemma, $\lim_{R\to\infty} \int_{\Gamma} \frac{e^{iz}}{z-ia}\, dz = 0$

$$\therefore \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{iz}}{z-ia}\, dz = 2\pi i \sum R^{+}$$

or $$\int_{-\infty}^{\infty} \frac{e^{iz}}{z-ia}\, dz = 2\pi i \sum R^{+}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(\cos x + i\sin x)(x+ia)}{(x-ia)(x+ia)}\, dx = 2\pi i e^{-a}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(x\cos x - a\sin x) + i(x\sin x + a\cos x)}{x^2+a^2}\, dx = i2\pi e^{-a}$$

Equating imaginary parts on both sides, we get,

$$\boxed{\int_{-\infty}^{\infty} \frac{x\sin x + a\cos x}{x^2+a^2}\, dx = 2\pi e^{-a}}$$

Hence, proved.

Q. 8. (d) Solve the following transportation problem.

	To			Available
From	2	7	4	5
	3	3	1	8
	5	4	7	7
	1	6	2	14
Required	7	9	18	34

Solution:

By Vogel's approximation the initial basic feasible solution having the transportation cost of ₹80 is obtained.

i.e. Initial basic feasible solution is
$\{3, 2, 0, 0, 0, 8, 0, 7, 0, 4, 0, 10\}$

	I	II	III
A	2 (3)	7 (2)	4 ×
B	3 ×	3 ×	1 (8)
C	5 ×	4 (7)	7 ×
D	1 (4)	6 ×	2 (10)

Also, no of basic cells = 6 = m+n−1.
∴ no degeneracy.

Now, using Mod-I method (u-v method) to find optimal basic feasible solution,

	I	II	III	
A	2 3 $+\varepsilon$ (0)	7 2 $-\varepsilon$ (0)	4 (−1)	u_1 (0)
B	3 (−3)	3 ε (2)	1 8 $-\varepsilon$ (0)	u_2 (−2)
C	5 (−6)	4 7 (0)	7 (−7)	u_3 (−3)
D	1 4 $-\varepsilon$ (0)	6 (0)	2 10 $+\varepsilon$ (0)	u_4 (−1)
	v_1 (2)	v_2 (7)	v_3 (3)	

Let $u_1=0$

$u_1+v_1=2 \Rightarrow v_1=2$

$u_1+v_2=7 \Rightarrow v_2=7$

$u_4+v_1=1 \Rightarrow u_4=-1$

$u_4+v_3=2 \Rightarrow v_3=3$

$u_2+v_3=1 \Rightarrow u_2=-2$

$u_3+v_2=4 \Rightarrow u_3=-3$

$\Delta_{13}=u_1+v_3-a_{13}$

$\Delta_{22}=u_2+v_2-a_{22}=2>0.$

only $\Delta_{22}>0$.

Hence, (2, 2) enters basis.

$\varepsilon=2$, leaving cell : (2, 1)

	I	II	III	
A	2 5 (0)	7 (−2)	4 (−1)	u_1 (2)
B	3 (−3)	3 2 (0)	1 6 (0)	u_2 (0)
C	6 (−4)	4 7 (0)	7 (−5)	u_3 (1)
D	1 2 (0)	6 (−2)	2 12 (0)	u_4 (1)
	v_1 (0)	v_2 (3)	v_3 (1)	

Let $u_2=0$.

Here, all the net evaluations are ≤ 0.
Hence, optimality is obtained.

At optimality,

A transports 5 units to I,

B transports 6 units to III, 2 units to II.

C transports 7 units to II.

D transports 2 units to I, 12 units to III.

Also, optimal cost of transportation =

$5(2)+2(3)+6(1)+7(4)+2(1)+12(2)$

$= \boxed{₹\ 76}$

Hence, the result.